New accurate methods for modelling of the continuity equation

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Abstract

A new efficient and accurate numerical method for solving the continuity equation is proposed. Aiming to fulfill two of the ten desirable properties, namely local mass conservation and computational efficiency, by Rasch and Williamson (1990) and Machenhauer et al. (2008). The new method is developed using cascade interpolation by Nair et al. and a locally mass conserving modification of the traditional semi-Lagrangian cubic interpolation scheme (LMCSL-scheme) by Kaas 2008.

The cascade interpolation reduces the number of interpolative operations needed, thereby enabling a scheme with higher computational efficiency. However, the scheme requires more memory than the LMCSL scheme due to the intermediate step in the cascade interpolation. The method has been tested on a slotted cylinder as described by e.g. Zerroukat et al. (2002) and we aim to implement it in a full scale chemical transport model.

Locally mass conserving semi-Lagrangian transport based on cascade interpolation is especially efficient when more (e.g. chemical) tracers are considered since they need not be recalculated for every specific tracer.

1. Theory

Cubic interpolation uses the four points closest to the true departure point to calculate the value of the arrival point in the next time step. In two dimensions this is called bicubic interpolation and uses 16 departure points.

Cascade interpolation splits a two dimensional problem into two one dimensional problems. This demands extra memory for the intermediate time step, but only 2×4 departure points.

Modified interpolation weights. Each departure point is weighted relative to its distance to the true departure point. In traditional schemes the weights for each point should sum to one but rarely does, therefore each weight is weighted relative to the total weighting of the point to achieve mass conservation.

Slotted cylinder and Cosine hill. To test the method the traditional Slotted cylinder and Cosine hill have been used. The slotted cylinder test is a 101 × 101 domain with a slotted cylinder rotating around the centre of the domain. The cylinder has an amplitude of 1 and the radius is 15 grid points, the slot is 6 grid points long and 3 grid points wide. The maximum Courant number for the method is approximately 3.27. The analytical solution is given by:

$$\psi(x,y,t)=\begin{cases}1 \text{ for } |\xi|\geq s_w/2,\ r\leq\sigma,\\ 1 \text{ for } \zeta\geq s_l-\sigma,\ r\leq\sigma,\\ 0 \text{ otherwise.}\end{cases}$$

With

$$\xi = x - x_c + \gamma\cdot cos(\omega t),$$
$$\zeta = y - y_c + \gamma\cdot sin(\omega t),$$
$$r = \sqrt{\xi^2+\zeta^2}.$$

The cosine hill is a 33 × 33 domain with a cosine hill with an amplitude of 100 units rotating around the center of the domain. The equations for this test is:

$$\psi(x,y,t)=\begin{cases}100\times[1+cos(\pi r/\sigma)]/2 & \text{for } r\leq\sigma,\\ 0 & \text{otherwise.}\end{cases}$$

2. Method

This method combines cubic semi-Lagrangian interpolation with cascade interpolation and modified interpolation weights.

The interpolation is performed in three steps, first the true departure points are found calculating the backward trajectory from the arrival point. Then the intermediate points are calculated - in these points the interpolation has been performed in the x-direction. Finally, the mass is transported in the other direction, in this case the y-direction.

Equations

$$w_{k-1,j}=\frac{-2\alpha+3\alpha^2-\alpha^3}{6},\quad w_{k,j}=\frac{2-\alpha-2\alpha^2-\alpha^3}{2},$$
$$w_{k+1,j}=\frac{2\alpha+\alpha^2-\alpha^3}{2},\quad w_{k+2,j}=\frac{-\alpha+\alpha^3}{6}.$$

$$\hat{w}_{k,j}=\frac{w_{k,j}(\alpha_k)}{\sum_{m=1}^{K} w_{m,k}(\alpha_m)}$$

$$\psi_{i,j}^{n+1x}=\hat{w}_{k-1,i}\psi_{i-k-1}^{n}+\hat{w}_{k,i}\psi_{i-k}^{n}+\hat{w}_{k+1,i}\psi_{i-k+1}^{n}+\hat{w}_{k+2,i}\psi_{i-k+2}^{n}.$$

$$\psi_{i,j}^{n+1}=\hat{w}_{l-1,j}\psi_{j-l-1}^{n+1x}+\hat{w}_{l,j}\psi_{j-l}^{n+1x}+\hat{w}_{l+1,j}\psi_{j-l+1}^{n+1x}+\hat{w}_{l+2,j}\psi_{j-l+2}^{n+1x}.$$

Error measures

$$l_1=\frac{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(|\phi_{i,j}-\psi_{i,j}|)}{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}|\psi_{i,j}|},$$

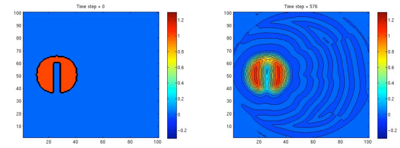
$$l_2=\frac{\sqrt{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(\phi_{i,j}-\psi_{i,j})^2}}{\sqrt{\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}(\psi_{i,j})^2}},$$

$$l_\infty=\frac{max\,[|\phi_{i,j}-\psi_{i,j}|]}{max\,[|\psi_{i,j}|]}.$$

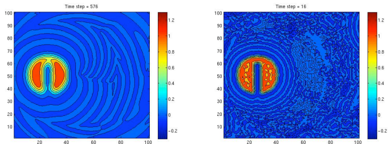
3. Results

The method has been tested on the slotted cylinder and the cosine hill test cases.

Slotted cylinder 6 revolutions standard time step.

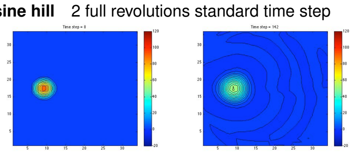


one revolution for shorter and longer time step respectively



Slotted cylinder			
Revolutions	l_1	l_2	l_∞
1	0.2612	0.0393	0.6169
2	0.3139	0.0697	0.6744
3	0.3537	0.0987	0.7063
4	0.3825	0.1214	0.7266
5	0.4052	0.1375	0.7409
6	0.4236	0.1479	0.7518

Cosine hill 2 full revolutions standard time step



Cosine hill			
Revolutions	l_1	l_2	l_∞
1	0.4791	0.0520	0.2996
2	0.6886	0.0821	0.4218

4. Discussion

The ten desirable properties, where boldface indicates which properties have been fulfilled.

- **Accuracy**
- **Stability**
- **Computational Efficiency**
- **Transportivity and Locality**
- Shape preservation
- **Conservation**
- **Consistency**
- Compatibility
- Preservation of Constancy
- **Preservation of linear correlations between constituents**

More than the two properties we aimed to fulfill also accuracy, stability, transportivity and locality, consistency and preservation of linear correlations between constituents have been fulfilled. The difference between a traditional semi-Lagrangian scheme and the method proposed here is conservation and consistency. The stability can be seen from the plots of the slotted cylinder with various time step, the method is not dependent on size of time step but the number of time step. Shape preservation could be achieved by application of a filter.

5. Conclusion

The aim of this project has been to develop a method which fulfills two of the desirable properties, namely Computational Efficiency and Conservation. We have developed a method combining classical semi-Lagrangian cubic interpolation with cascade interpolation and weighted weights. This was tested using a slotted cylinder and a cosine hill.

As can be seen from the test results the method preserves the shape of the slotted cylinder well. However, some over- and undershooting occurs. A filter which removes this will be added. The method conserves mass to machine accuracy - locally and globally.

The computational efficiency has not been tested explicitly, but in a case with several tracers it is possible to use the weighted weights for all tracers. This makes the method developed here very efficient.

By comparing this method to a similar method without weighted weights it can be seen that the weighted weights impose conservation of mass in the model. Also, it is clear that the accuracy of the model depends on the number of time steps not the length of the time steps. This is particularly useful in climate models.

6. Acknowledgement

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