Reply

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1. Discussion

Smolarkiewicz's comment on my paper (Bott 1989; hereafter referred to as B89) contains several criticisms with which I do not agree. Since his terminology partially deviates from B89, some clarifications are first necessary. The quantities $F_{i\pm1/2}$ of the Comment which Smolarkiewicz calls transport fluxes correspond to the integrals $I_l^{\pm}(c_{j+1/2})$ as defined in (7) and (8) of B89 (for brevity, hereafter referred to as B7 and B8). Note that the physical dimension of the fluxes should be $[u\psi]$, where u is the transport velocity (for example, given in m s^{-1}). Hence, fluxes are obtained after multiplication of F and $\tilde{F}$ by $\Delta x/\Delta t$ as in (B4), (B9), (B12) and (B13). Furthermore, although Smolarkiewicz suggests that his equation (S2) corresponds to the first step of my advection procedure as expressed in (B12), this is not the case. Consider, for example, the situation at the right boundary of a given ψ-distribution with $\psi_{i-1} \gg \psi_i > \psi_{i+1} > 0$, $\psi_k = 0$, $\forall k > i + 1$ and $u > 0$ everywhere. Fitting the ψ-distribution in grid box i with higher order polynomials ($l \geq 2$) may result in $F_{i+1/2} < 0$ and $F_{i+1/2}|_{\alpha=1} > 0$. Hence, in contrast to (B12), application of (S2) now yields a nonphysical flux from box $i + 1$ to i, i.e., in the opposite flow direction. Therefore, the use of the operators $[\ \ ]^+$ and $[\ \ ]^-$ throughout the Comment is misleading.

Smolarkiewicz correctly states that my advection procedure proceeds in two steps. In the first step the transport fluxes obtained by a given advection scheme are multiplied by appropriate weighting factors (flux weighting) before in the second step the modified fluxes are nonlinearly limited by upper and lower values (flux limitation). He has not, however, correctly recognized the importance and the role that each of these steps plays in the advection procedure.

The main purpose of B89 is to obtain a given advection algorithm positive definite. In addition to the positive definiteness it is also very important to reduce phase and amplitude errors as much as possible. Therefore, in contrast to Smolarkiewicz's assertion, positive definiteness is achieved by carefully considering physical reasons and not by imposing arbitrary limits on the advective fluxes. This is the reason why the fluxes are weighted first before they are appropriately bounded. In order to clarify in more detail the role of the flux weighting, I will briefly repeat now the logic of my derivation which has been distorted by Smolarkiewicz.

In section 2 of B89, the special case with $|c_{j+1/2}| = 1$ has only been considered as an example that clearly demonstrates why and how flux weighting is performed. The reason for introducing this procedure is to obtain in a grid box a better description of the ψ-distribution than with the original fitting, i.e., the integrated flux form of Tremback et al. (1987). Note that with this form the polynomials $\psi_{j,l}(x')$ are constructed from the requirement that at $x_j, x_{j\pm1}, \ldots$ the values of $\psi_{j,l}(x')$ agree with $\psi_j, \psi_{j\pm1}, \ldots$. Due to this requirement, however, the area covered by $\psi_{j,l}(x')$ is in grid box j generally not given by $\psi_j\Delta x$ (for $l \geq 2$). This means that the polynomial fitting is not area-preserving. Since the polynomials are introduced to represent in a grid box the local distribution of the given ψ-content, it is physically more meaningful to construct the curves with the requirement of area preservation instead of demanding agreement between $\psi_{j,l}(x')$ and $\psi_j, \psi_{j\pm1}, \ldots$ at $x_j, x_{j\pm1}, \ldots$. Performing the flux weighting is equivalent to the parallel shift of $\psi_{j,l}(x')$ in such a way that area preservation is achieved. Hence, in each grid box the polynomials are constructed by means of two physical properties of the ψ-field, i.e., the trend of the curve and the total ψ-content of the box.

Smolarkiewicz does not recognize that flux-weighting is indeed related to the positive definiteness of the algorithm even though it does not make the scheme positive definite. (The latter has never been claimed in B89.) From (B12) it is clear that the advective fluxes are linear functions of the weighting factors. The effect of the linear weighting process on positive definiteness of the algorithm is to weaken the importance of the following nonlinear flux limitation. This is illustrated in the example with large Courant numbers $c_{j+1/2} \sim 1$ and $I_{l,j} > \psi_j$ in (B11). Without flux-weighting the scheme may become only positive definite when flux

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limitation is carried out, but performing the weighting procedure already maintains positive definiteness so that the fluxes are unaffected by the following flux limitation. Hence, due to the careful design of the polynomials by means of physical arguments, the nonlinear flux limitation may be reduced or become unnecessary.

This is why flux weighting yields a distinct improvement of the results by reducing phase speed errors. Comparing Figs. 1 and 2 of the Comment, one can see that in contrast to Smolarkiewicz's opinion, the improvements are quite apparent, particularly since these are achieved in a very simple and numerically efficient manner. In order to elucidate more clearly the effect of flux weighting, I repeated the one-dimensional advection experiments with a single Fourier mode of wavelength $4\Delta x$ for model version $l = 4$ but without performing flux weighting. Results are depicted in Fig. 1. For comparison the results obtained with the original scheme, i.e., with inclusion of flux weighting, are shown in Fig. 2. As can be clearly seen from these figures, phase speed errors decrease distinctly when flux weighting is carried out. Furthermore, omitting this procedure yields an increase of amplitude errors which becomes very pronounced for large Courant numbers. Comparing Fig. 1 with the corresponding results of the advection scheme of Smolarkiewicz (1983) (see Fig. 3), however, clarifies that in the proposed method the most important improvement is due to the fitting of the ψ-distribution with higher order polynomials. This holds particularly for small Courant numbers $c = 0.25$ where the scheme of Smolarkiewicz, contrary to his opinion, produces comparatively large phase and amplitude errors. Similar considerations also hold for the Leapfrog algorithm. (Compare Figs. 5 and 6 of the comment with Figs. 3d–f of B89.)

Although the logic of Smolarkiewicz's derivation differs from mine, he arrives at a similar result (his "upstream renormalization"). In his approach, however, the question of how the term $\psi_{i+1/2}$ may be appropriately chosen remains unanswered. A simple method to overcome this problem is to draw from (B12) $\psi_{i+1/2} = F_{i+1/2}|_{\alpha=1}$ (using the nomenclature of the comment). The results obtained with $(0.88)^{-1} \times F_{i+1/2}|_{\alpha=0.88}$ (Fig. 4 of the comment) are seemingly good. Like the introduction of the "correction coefficient" Sc $\neq$ 1 in Smolarkiewicz (1983), however, this formula is derived without any mathematical or physical arguments. Thus, the encouraging result obtained in a particular situation does not imply that the same quality of the results will be obtained in the general case.

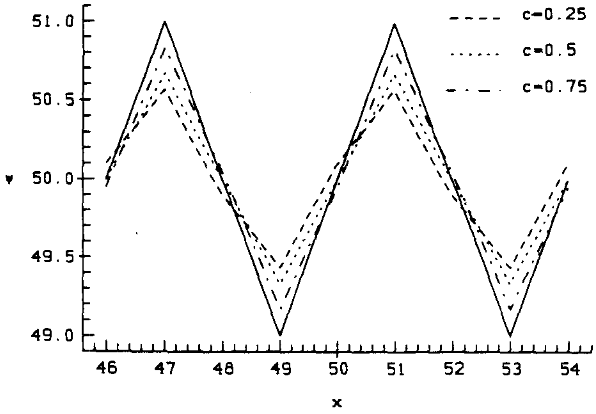

FIG. 2. As in Fig. 1, but including flux weighting.

The formalism for sign preservation as presented in section 3 of the Comment repeats my own algorithm except that the importance of the flux weighting procedure has not been recognized. Of course, extending the proposed method to the multidimensional case is possible in principle. [Note that Eq. (S10) may only be used after carefully redefining the operators [ ]$^{\pm}$.] As pointed out in B89 and in Tremback et al. (1987),

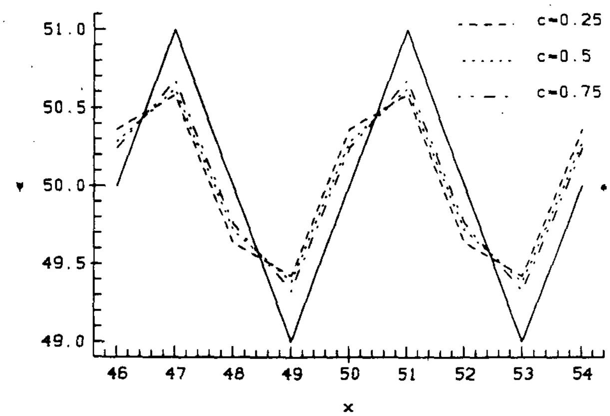

FIG. 1. Results from the one-dimensional stability test with a single Fourier mode of wavelength $4\Delta x$ for model version $l = 4$ without flux weighting. Solid (dashed) lines represent the analytical solution (numerical solution for various Courant numbers c).

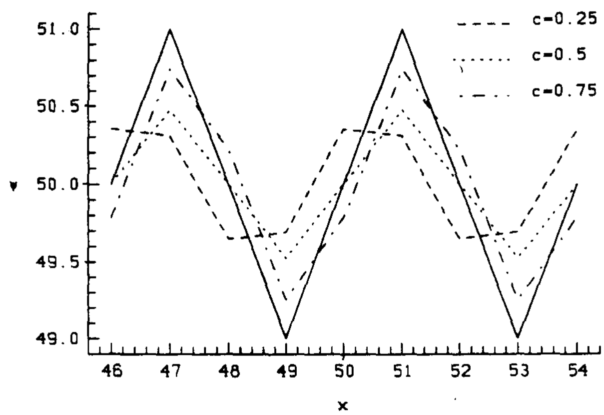

FIG. 3. As in Fig. 1, but obtained with the Smolarkiewicz's 1983 advection scheme.

TABLE 1. Coefficients $a_{j,k}$ for the $l = 2$ and $l = 4$ versions of the area-preserving flux form.

	$l = 2$	$l = 4$
$a_{j,0}$	$-\frac{1}{24}(\psi_{j+1} - 26\psi_j + \psi_{j-1})$	$\frac{1}{1920}(9\psi_{j+2} - 116\psi_{j+1} + 2134\psi_j - 116\psi_{j-1} + 9\psi_{j-2})$
$a_{j,1}$	$\frac{1}{2}(\psi_{j+1} - \psi_{j-1})$	$\frac{1}{48}(-5\psi_{j+2} + 34\psi_{j+1} - 34\psi_{j-1} + 5\psi_{j-2})$
$a_{j,2}$	$\frac{1}{2}(\psi_{j+1} - 2\psi_j + \psi_{j-1})$	$\frac{1}{48}(-3\psi_{j+2} + 36\psi_{j+1} - 66\psi_j + 36\psi_{j-1} - 3\psi_{j-2})$
$a_{j,3}$	—	$\frac{1}{12}(\psi_{j+2} - 2\psi_{j+1} + 2\psi_{j-1} - \psi_{j-2})$
$a_{j,4}$	—	$\frac{1}{24}(\psi_{j+2} - 4\psi_{j+1} + 6\psi_j - 4\psi_{j-1} + \psi_{j-2})$

though, the resulting code would become numerically expensive. In particular, this holds for the model versions $l = 2$, 4 which B89 recommends due to their small phase and amplitude errors. Hence, I intentionally omitted the derivation of the corresponding formulas.

As mentioned above, performing the flux weighting is equivalent to making the given polynomials area-preserving. The resulting improvements suggest that it may be physically reasonable to construct directly the polynomials with the requirement of area preservation. This is easily done by solving at grid point j the linear system of $l + 1$ equations

$$\psi_i \Delta x_i = \int_{x_{i-1/2}}^{x_{i+1/2}} \sum_{k=0}^{l} a_{j,k} x'^k dx', \; i = j, j \pm 1, \cdots.$$

Here the term $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ represents the more general case with variable grid spacing. Nevertheless, for the sake of simplicity, Table 1 only depicts the coefficients $a_{j,k}$ which are obtained for polynomials of order $l = 2$ and $l = 4$ in the special situation with constant Δx. Substitution of these coefficients into (B7)–(B11) yields together with (B12)–(B14) the area-preserving flux form. Comparing this form with the constant grid flux form of Tremback et al. (1987) reveals that in the special case with constant grid meshes both model versions are identical (when flux limitation is omitted). Due to the fact that the area-preserving flux form is extenable to variable grid spacings, however, the name "constant grid flux form" is somewhat misleading and has been replaced.

Figure 4 depicts the results of the one-dimensional Fourier tests for the $l = 4$ version of the area-preserving flux form, elucidating that the new algorithm yields a further strong reduction of phase and amplitude errors. This holds particularly for the lower Courant numbers $c \leqslant 0.5$ where the other presented methods are less satisfactory (see Figs. 2 and 3). These encouraging results suggest that in the multidimensional case (i.e., when time splitting is applied) the area-preserving flux form may also be superior to other known algorithms. This is demonstrated in Fig. 5 depicting the results of the rotational flow field test. After six revolutions the maximum of the ψ-distribution has only decreased to 89% of its initial value in comparison to 86% for the original $l = 4$ version (see Table 2 of B89). Both model versions need similar computational effort which is less than the fourfold time as required by the upstream method. This elucidates that, in contrast to the suggestions of Smolarkiewicz, the flux weighting procedure is only of minor importance for the computational efficiency of the scheme. Comparison of Fig. 5 of this reply and Figs. 3d–f of B89 with the corresponding Figs. 2–15 of Smolarkiewicz (1983) clearly demonstrates the distinct superiority of my scheme over the hybrid schemes (Clark–Hall, SAHS, FCT) and over all versions of the Smolarkiewicz algorithm. This holds for the numerical accuracy (phase and amplitude errors) as well as for the computational efficiency [compare Table 1 of Smolarkiewicz (1983) with Table 2 of B89].

2. Conclusion

The main purpose of my advection scheme is to be positive definite and computationally very efficient. At

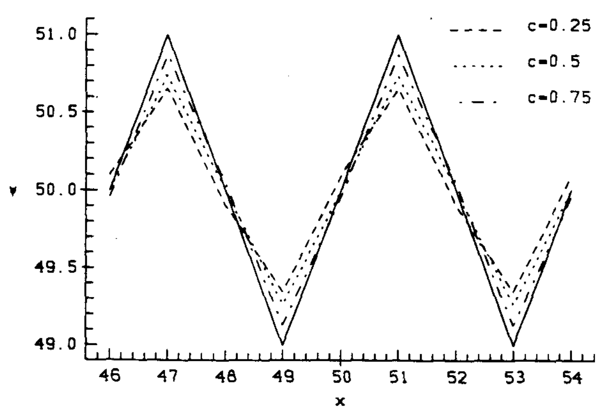

FIG. 4. As in Fig. 1, but with model version $l = 4$ of the area-preserving flux form.

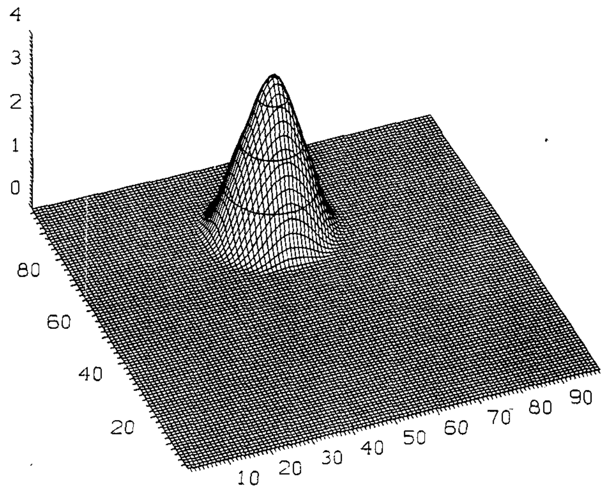

FIG. 5. Results from the rotational flow field experiment after six full rotations for the $l = 4$ version of the area-preserving flux form.

the same time the model should produce only small phase and amplitude errors. The resulting algorithm satisfies all these conditions, and furthermore has the advantage to be extendable to other known advection schemes.

In several points I disagree with Smolarkiewicz's interpretation of my method. Particularly, he underestimates the importance of the flux weighting process. In contrast to his assertion, I clearly point out in B89 the different roles which each of the two steps play in the procedure, i.e., the reduction of phase speed errors in the first step and the positive definiteness of the scheme obtained in the second step. The formalism presented in section 3 of Smolarkiewicz to obtain sign preservation is exactly the same as my procedure and, therefore, its presentation in the comment is redundant. The necessity of extending the algorithm to multidimensions seems rather questionable with respect to the required computational efficiency.

The proposed advection scheme provides a very simple and numerically efficient method to strongly reduce phase and amplitude errors. This has been demonstrated by the area-preserving flux form in B89 and in this Reply. This form is applicable to variable grid spacings and reduces for constant Δx to the constant grid flux form of Tremback et al. (1987).

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