A Positive Definite Advection Scheme Obtained by Nonlinear Renormalization of the Advective Fluxes

ANDREAS BOTT
Meteorological Institute, Johannes Gutenberg University, Mainz, FRG

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ABSTRACT

A new method is developed to obtain a conservative and positive definite advection scheme that produces only small numerical diffusion. Advective fluxes are computed utilizing the integrated flux form of Tremback et al. These fluxes are normalized and then limited by upper and lower values. The resulting advection equation is numerically solved by means of the usual upstream procedure. The proposed treatment is not restricted to the integrated flux form but may also be applied to other known advection algorithms which are formulated in terms of advective fluxes.

Different numerical tests are presented illustrating that the proposed scheme strongly reduces numerical diffusion and simultaneously requires only small computational effort. For Courant numbers with absolute values not exceeding one, the scheme preserves numerical stability except in strong deformational flow fields where slight instabilities may occur.

1. Introduction

Numerical modeling of atmospheric processes requires the solution of the continuity equation describing the transport of a particular quantity in a given flow field. The literature provides a wide range of numerical procedures to solve this equation, e.g., Lax and Wendroff (1964), Crowley (1968), Book et al. (1975). One of the most popular advection schemes is the first upwind differencing or upstream method as described in great detail by various authors. The method is explicit, forward-in-time, and therefore numerically very efficient. Furthermore, the scheme is positive definite, i.e., positive defined quantities remain positive through the whole advection process. This property is of particular interest when, for example, the transport of atmospheric pollutants or the diffusional growth of cloud droplets is treated, since nonphysical results with negative concentrations are avoided. However, the algorithm is accurate only to first order in space and time and, therefore, produces strong numerical diffusion.

Crowley (1968) introduced the method of polynomial fitting to develop higher order accurate schemes. His advection procedure has been studied and improved by some authors (e.g., Petschek and Libersky 1975; Smolarkiewicz 1982; Schlesinger 1985). Based on Crowley's second-order scheme Smolarkiewicz (1983, 1984) developed an advection procedure which is conservative as well as positive definite, but simultaneously produces less numerical diffusion than the upstream method. The algorithm consists in the introduction of corrective advection fluxes which reduce the truncation error produced by the upstream solution. More recently, Tremback et al. (1987) extended Crowley's original version to polynomials of orders up to 10 achieving strong reduction of numerical diffusion. Unfortunately, these schemes are not positive definite and, thus, may sometimes lead to undesirable results by producing negative values of the transported quantity.

This paper presents a new method to obtain a positive definite version of the integrated flux form of Tremback et al. (1987). Tremback et al.'s advective fluxes are first normalized with a view to reducing the phase speed errors. Negative values of the transported quantity are then suppressed by nonlinearly limiting the normalized fluxes. The proposed treatment is formulated in such a manner that it is not restricted to the integrated flux method but may also be extended to other known advection procedures which can be formulated in flux form.

One-dimensional numerical tests with single Fourier modes are presented elucidating that the positive definite version of the integrated flux form produces only small phase- and amplitude errors. These errors clearly decrease with increasing order of polynomials. Two-dimensional advection experiments substantiate these findings, i.e., numerical results are considerably improved when model versions with higher ordered polynomials are utilized.

In uniform flow the proposed advection scheme preserves numerical stability for all Courant numbers

Corresponding author address: Dr. Andreas Bott, Meteorological Institute, Johannes Gutenberg Universitat, D 6500 Mainz Postfach 39 80, Federal Republic of Germany.

with absolute values not exceeding one. Even in strong deformational flow fields only slight numerical instabilities are observed.

2. Theory

The continuity equation describing the transport of a nondiffusive quantity $\psi(x, y, z, t)$ in a flow field is given by

$$\frac{\partial \psi}{\partial t} = -\nabla \cdot (\mathbf{v}\psi). \quad (1)$$

For simplicity the proposed numerical solution of (1) will be derived only for the one-dimensional case

$$\frac{\partial \psi}{\partial t} = -\frac{\partial u\psi}{\partial x}. \quad (2)$$

Extension to higher dimensions is straightforward when the time splitting method is used, and numerical results for two-dimensional flow fields will be presented in the next section. With the assumption of constant grid spacing Δx and time increments Δt the finite difference flux form of (2) reads

$$\psi_j^{n+1} = \psi_j^n - \frac{\Delta t}{\Delta x}[F_{j+1/2}^n - F_{j-1/2}^n], \quad (3)$$

where ψ_j^n is the value of ψ at grid point j after n time steps and $F_{j+1/2}^n$, $F_{j-1/2}^n$ are the ψ-fluxes through the right and left boundary of the grid box, respectively.

Using the upstream method to solve (2) yields the ψ-flux through the right boundary

$$F_{j+1/2}^n = \frac{\Delta x}{\Delta t}[c_j^+\psi_j^n - c_j^-\psi_{j+1}^n]. \quad (4)$$

Here, the abbreviations $c_j^\pm = \pm(c_{j+1/2}^n \pm |c_{j+1/2}^n|)/2$ have been introduced and the term $c_{j+1/2}^n$ is the Courant number defined by $c_{j+1/2}^n = u_{j+1/2}^n\Delta t/\Delta x$. In order to simplify the notation the superscript n will be omitted henceforth except where confusion is possible. Numerical stability of the scheme is ensured if the Courant-Friedrich-Levy (CFL) criterion

$$c_j^+ + c_{j-1}^- \leq 1 \quad (5)$$

is fullfilled at each time step.

As previously stated the upstream method has the important properties of being positive definite and conservative. Unfortunately, it is only first-order accurate in space and time, thus producing strong numerical diffusion. This makes the scheme rather useless, particularly in cases of strong spatial gradients of ψ. In the following an advection scheme is proposed that greatly reduces numerical diffusion without losing the conservative and positive definite properties of the upstream method.

The large numerical diffusion generated by the upstream scheme is due to the poor representation of ψ by a step function with constant values in each grid box. Crowley (1968) and Tremback et al. (1987) used the method of polynomial fitting to obtain a better description of the ψ-field. Following their concept it is assumed that within grid box j the distribution of ψ is given by a polynomial of order l,

$$\psi_{j,l}(x') = \sum_{k=0}^{l} a_{j,k}x'^k \quad (6)$$

with $x' = (x - x_j)/\Delta x$ and $-½ \leq x' \leq ½$. The coefficients $a_{j,k}$ are functions of $(l + 1)$ ψ-values and may be obtained by interpolating the ψ-curve with the aid of neighboring grid points. Values of $a_{j,k}$ for $l = 0, \cdots, 4$ are summarized in Table 1.

While even ordered polynomials are constructed using the same number of grid points to the left and right of x_j, an extra point x_i ($i = j \pm (l + 1)/2$) is needed for odd ordered polynomials. Hence, in Table 1 for $l = 1$ and 3 two different sets of coefficients $a_{j,k}$ for $i = j + (l + 1)/2$ (1a, 3a) and $i = j - (l + 1)/2$ (1b, 3b) are specified. As will be expected, the asymmetric shape of odd ordered polynomials has some influence on the numerical results, especially for $l = 1$.

Defining and evaluating the integrals

$$I_l^+(c_{j+1/2}) = \int_{1/2-c_j^+}^{1/2} \psi_{j,l}(x')dx'$$

$$= \sum_{k=0}^{l} \frac{a_{j,k}}{(k+1)2^{k+1}}[1 - (1 - 2c_j^+)^{k+1}] \quad (7)$$

TABLE 1. Coefficients $a_{j,k}$ of Eq. (5) for various orders l of polynomials. For $l = 1$ and $l = 3$ two different sets of coefficients are represented, see also text.

l	$a_{j,0}$	$a_{j,1}$	$a_{j,2}$	$a_{j,3}$	$a_{j,4}$
0	ψ_j	—	—	—	—
1a	ψ_j	$\psi_{j+1} - \psi_j$	—	—	—
1b	ψ_j	$\psi_j - \psi_{j-1}$	—	—	—
2	ψ_j	$\frac{1}{2}(\psi_{j+1} - \psi_{j-1})$	$\frac{1}{2}(\psi_{j+1} - 2\psi_j + \psi_{j-1})$	—	—
3a	ψ_j	$\frac{1}{6}(-\psi_{j+2} + 6\psi_{j+1} - 3\psi_j - 2\psi_{j-1})$	$\frac{1}{2}(\psi_{j+1} - 2\psi_j + \psi_{j-1})$	$\frac{1}{6}(\psi_{j+2} - 3\psi_{j+1} + 3\psi_j - \psi_{j-1})$	—
3b	ψ_j	$\frac{1}{6}(2\psi_{j+1} + 3\psi_j - 6\psi_{j-1} + \psi_{j-2})$	$\frac{1}{2}(\psi_{j+1} - 2\psi_j + \psi_{j-1})$	$\frac{1}{6}(\psi_{j+1} - 3\psi_j + 3\psi_{j-1} - \psi_{j-2})$	—
4	ψ_j	$\frac{1}{12}(-\psi_{j+2} + 8\psi_{j+1} - 8\psi_{j-1} + \psi_{j-2})$	$\frac{1}{24}(-\psi_{j+2} + 16\psi_{j+1} - 30\psi_j + 16\psi_{j-1} - \psi_{j-2})$	$\frac{1}{12}(\psi_{j+2} - 2\psi_{j+1} + 2\psi_{j-1} - \psi_{j-2})$	$\frac{1}{24}(\psi_{j+2} - 4\psi_{j+1} + 6\psi_j - 4\psi_{j-1} + \psi_{j-2})$

$$I_l^-(c_{j+1/2}) = \int_{-1/2}^{-1/2+c_j^-} \psi_{j+1,l}(x')dx'$$
$$= \sum_{k=0}^{l} \frac{a_{j+1,k}}{(k+1)2^{k+1}} (-1)^k[1-(1-2c_j^-)^{k+1}] \quad (8)$$

yields the ψ-flux through the right boundary of grid box j

$$F_{j+1/2} = \frac{\Delta x}{\Delta t} [I_l^+(c_{j+1/2}) - I_l^-(c_{j+1/2})]. \quad (9)$$

In this form the algorithm is conservative and agrees with the integrated flux form of Tremback et al. (1987). (Note the different signs in the definitions of the advective fluxes and in (3) compared with the corresponding equation (6) of Tremback et al.) However, the scheme still lacks positive definiteness. A sufficient condition for this is

$$0 \leq I_l^+(c_{j+1/2}) + I_l^-(c_{j-1/2}) \leq \psi_j. \quad (10)$$

Hence, the total outflow of a grid volume is always positive and limited by ψ_j. Inspection of (7) and (8) shows that this condition holds for $l = 0$ since in this case (9) reduces to the upstream version (4). But for $l > 0$ the integrals may not necessarily comply with (10). For example, if $l \geq 2$ and $|c_{j+1/2}| = 1$ the von Neumann stability analysis yields amplification factors and phase speed ratios lower than one, (see Tables 3 and 4 of Tremback et al. 1987), indicating that the exact solution of the advection equation is not reproduced. This is due to the fact that for $|c_{j+1/2}| = 1$ the analytical solution is only obtained if in a grid box the area covered by $\psi_{j,l}(x')$ is given by $\psi_j\Delta x$. For $l = 0$, 1 this is always the case, but polynomials of order $l \geq 2$ do in general not satisfy this condition. To account for occurring differences it is, therefore, appropriate to multiply $\psi_{j,l}(x')$ by a weighting factor $\psi_j/I_{l,j}$ with $I_{l,j} = I_l^\pm(c_{j\pm1/2} = \pm1)$ and

$$I_{l,j} = \int_{-1/2}^{1/2} \psi_{j,l}(x')dx'$$
$$= \sum_{k=0}^{l} \frac{a_{j,k}}{(k+1)2^{k+1}} [(-1)^k + 1]. \quad (11)$$

Now the total area covered by the weighted polynomial is in grid box j given by $\psi_j\Delta x$, and the advection scheme yields exact results for $|c_{j+1/2}| = 1$ if the integrals on the right-hand side of (9) are multiplied by the appropriate weighting factors. This yields the advective flux as

$$F_{j+1/2} = \frac{\Delta x}{\Delta t}\left[\frac{I_l^+(c_{j+1/2})}{I_{l,j}}\psi_j - \frac{I_l^-(c_{j+1/2})}{I_{l,j+1}}\psi_{j+1}\right]. \quad (12)$$

Finally, positive definiteness of the algorithm is achieved if the advective fluxes, in accordance with (9) and (10), are limited by upper and lower values. Particularly, (for positive velocities) $F_{j+1/2}$ is set equal 0 whenever $I_l^+(c_{j+1/2}) < 0$. Furthermore, the flux must be limited by $\psi_j\Delta x/\Delta t$ if $I_l^+(c_{j+1/2}) > I_{l,j}$. Corresponding flux limitations must be introduced for negative velocities. Combining all necessary restrictions the advection scheme may be effectively coded by defining

$$F_{j+1/2} = \frac{\Delta x}{\Delta t}\left[\frac{i_{l,j+1/2}^+}{i_{l,j}}\psi_j - \frac{i_{l,j+1/2}^-}{i_{l,j+1}}\psi_{j+1}\right] \quad (13)$$

with

$$i_{l,j+1/2}^+ = \max(0, I_l^+(c_{j+1/2}))$$
$$i_{l,j+1/2}^- = \max(0, I_l^-(c_{j+1/2}))$$
$$i_{l,j} = \max(I_{l,j}, i_{l,j+1/2}^+ + i_{l,j-1/2}^- + \epsilon). \quad (14)$$

(The small term $\epsilon > 0$ has only been introduced to avoid the numerical unstable situation with $i_{l,j} = 0$.) The last equation of (14) takes also into account the case with $c_{j+1/2} > 0$ and $c_{j-1/2} < 0$. In this situation it must be considered that the outflow in grid box j takes place at the boundaries $j - 1/2$ and $j + 1/2$. Hence, the sum of both fluxes must be limited by $\psi_j\Delta x/\Delta t$.

Comparison of (13) with (4) shows that the terms $c_j^\pm$ appearing in the upstream solution have now been replaced by quotients which, due to (14), are restricted to the same limits as the $c_j^\pm$ in (5). Hence, interpreting these quotients as effective Courant numbers it is obvious that the advection algorithm is conservative and positive definite for all orders of polynomials. Nevertheless, at each time step the CFL criterion (5) must not be violated by the velocity field since the integrals in (7) and (8) are only defined for Courant numbers not exceeding 1.

From the equations derived above it can be seen that the presented formalism for obtaining a weakly diffusive conservative positive-definite advection scheme is not restricted to the integrated flux method but may also be extended to other known algorithms in which advective fluxes are calculated and where ψ_j^{n+1} is given by (3). Therefore, the integrated flux form may be considered as an example which demonstrates how the formalism works. In general, our procedure has two steps. First, the advective fluxes are multiplied by an appropriate weighting factor which is given as the ratio of the analytical flux divided by the advective flux of the particular scheme where both fluxes have been evaluated for $|c_{j+1/2}| = 1$. In a second step, positive definiteness of the algorithm is achieved by limiting the weighted fluxes as described in (14).

For instance, it would be straightforward to obtain a positive definite version of the constant grid flux form of Tremback et al. Note that with this scheme only the flux limitation (14) has to be carried out because the weighting factors are given by 1 for all orders of polynomials. Thus, the algorithm represents a special case of the formalism derived above. Furthermore, in contrast to the integrated flux form, the method is restricted to constant grid spaces.

Due to the possibility of using different advection schemes to construct a positive definite algorithm the resulting accuracy of the scheme depends on the particular method which has been chosen. In the example given in this paper model version $l = 0$ agrees with the upstream scheme and is, therefore, first-order accurate in space and time. Without flux limitation version $l = 1$a would be identical to the ORD = 2 version of the integrated flux form of Tremback et al., thus being second-order accurate in space. As already pointed out by these authors a further increase of the order of polynomials yields, however, not necessarily higher order accurate model versions since the integrated flux form does no more agree with the advective form. But only the latter produces by construction always the same order of accuracy as the utilized Lagrangian polynomial. Hence, it is not guaranteed that for increasing l the results must always be improved.

By comparing the model for $l \geq 2$ with Taylor series expansions an error analysis would further be complicated due to the introduction of the weighting factors and the nonlinear limitation of the advective fluxes carried out in (14). Similar considerations hold also for the constant grid flux form. Although this method produces the same orders of accuracy as the advective form the positive definite model version would not yield the same errors due to the nonlinear flux-normalization which must be carried out.

The accuracy of the presented scheme (or another one) may, therefore, better be demonstrated in numerical advection experiments. In the following section some examples of such experiments are presented for the integrated flux form. All numerical results elucidate that for $l = 0, \cdots, 4$ numerical diffusion produced by the scheme clearly decreases with increasing order of polynomials. Numerous different advection experiments have been additionally carried out which are not presented in this paper. All these tests substantiated a distinct improvement of the results for higher ordered polynomials (where $l \leq 4$).

3. Numerical results

a. One-dimensional experiments with single Fourier modes

In order to demonstrate numerically the accuracy of the proposed scheme one-dimensional experiments with single Fourier modes of wavelengths $4\Delta x$ and $8\Delta x$ have been performed according to Smolarkiewicz and Clark (1986). These tests give qualitatively a good insight into the amplitude and phase errors produced by different model versions ($l = 1, \cdots, 4$). Numerical results for constant Courant numbers $c = 0.25$, 0.5, and 0.75 are depicted in Figs. 1 and 2 after $3/c$ time steps. The analytical solution is also represented in each figure. Since the algorithm is restricted to positive ψ values, a particular Fourier mode has been shifted by a constant $\psi_0 = 50$ before the advection is calculated.

Obviously, model version 1b yields the worst agreement with the analytical solution, while version 1a produces lower amplitude and phase errors. This is due to the fact that at $j + ½$ the ψ-field is better represented as function of the two neighboring grid values ψ_j and ψ_{j+1} than as function of ψ_j and ψ_{j-1}. Increasing the order of polynomials further improves the quality of the results, but the rate of improvement is reduced for larger l. Model versions $l = 2$, 4 produce lower phase errors than versions ($l = 1$, 3) reflecting the findings of Tremback et al. (1987). Depending on the value of the chosen Courant number the amplitude damping of versions $l = 1$a, b and 3a, b is partly lower or exceeds the damping obtained with $l = 2$ and 4, respectively. The difference between model version 3a and 3b is not as large as in the case of 1a and 1b because for $l = 3$ the polynomial $\psi_{j,l}(x')$ only weakly depends on the more distant ψ-field. As was to be expected a particular model version yields more accurate results for longer wavelengths. Strongest damping occurs at wavelength $2\Delta x$ (not shown). Here, the distributions qualitatively resemble those obtained by Smolarkiewicz and Clark (1986). Further calculations with different values of ψ_0 (not shown) demonstrated that numerical results are only barely affected by this term.

Summarizing above findings it can be concluded that model versions 2 and 4 may be preferred because they produce always lower phase errors while amplitude damping is not necessarily largest. In cases of weak spatial gradients of ψ (long wavelengths) the model with $l = 2$ seems to produce sufficiently acceptable results. In situations with strong spatial gradients of ψ, however, the fourth-order polynomial should be utilized to fit the ψ-distribution more appropriately.

In general it is possible to obtain a positive definite version of the integrated flux form without introducing the weighting factors. In this case the integrals $I_l^{\pm}$ must be directly limited in such a way that equation (10) is fulfilled. In order to obtain an impression of the accuracy of the resulting scheme the above described advection experiments have been repeated utilizing this version. Numerical results which are not shown depicted a considerable increase of phase errors, particularly for the higher ordered polynomials. Obviously, these findings indicate that it is necessary to multiply the advective fluxes by appropriate weighting factors before flux limitation is carried out.

b. Rotational flow field test

In order to facilitate a comparison between the proposed and already existing positive definite advection schemes, the two-dimensional tests carried out by Smolarkiewicz (1982) have been repeated. In the first numerical experiment a cone with base radius $15\Delta x$ and maximum height $\psi_{max} = 3.87$ at $(x, y) = (50\Delta x, 75\Delta y)$ is located in a two-dimensional domain of 100 by 100 grid points with $\Delta x = \Delta y = 1$; see Fig. 3a. The

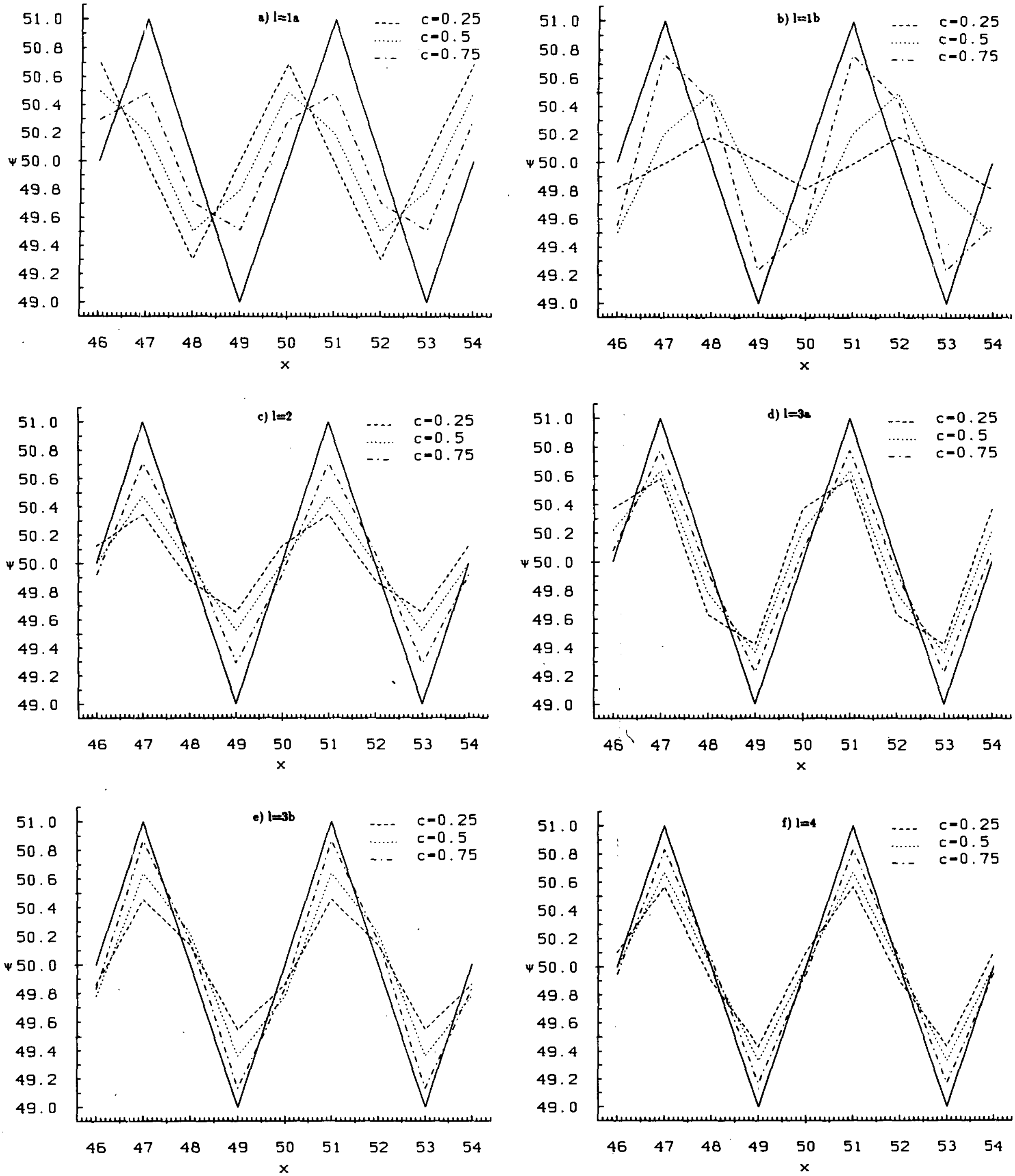

FIG. 1. Results from the one-dimensional stability test with a single Fourier mode of wavelength $4\Delta x$ for different model versions. Solid (dashed) lines represent the analytical solution (numerical solution for various Courant numbers c).

cone rotates with constant angular velocity $\omega = 0.1$ counterclockwise around the point $(x_0, y_0) = (50, 50)$. The integration is carried out with $\Delta t = 0.1$ utilizing the time splitting method. As already pointed out by Tremback et al. (1987) this integration procedure requires less computational effort than extending the method of polynomial fitting to higher dimensions. Results for different orders l of polynomials are shown

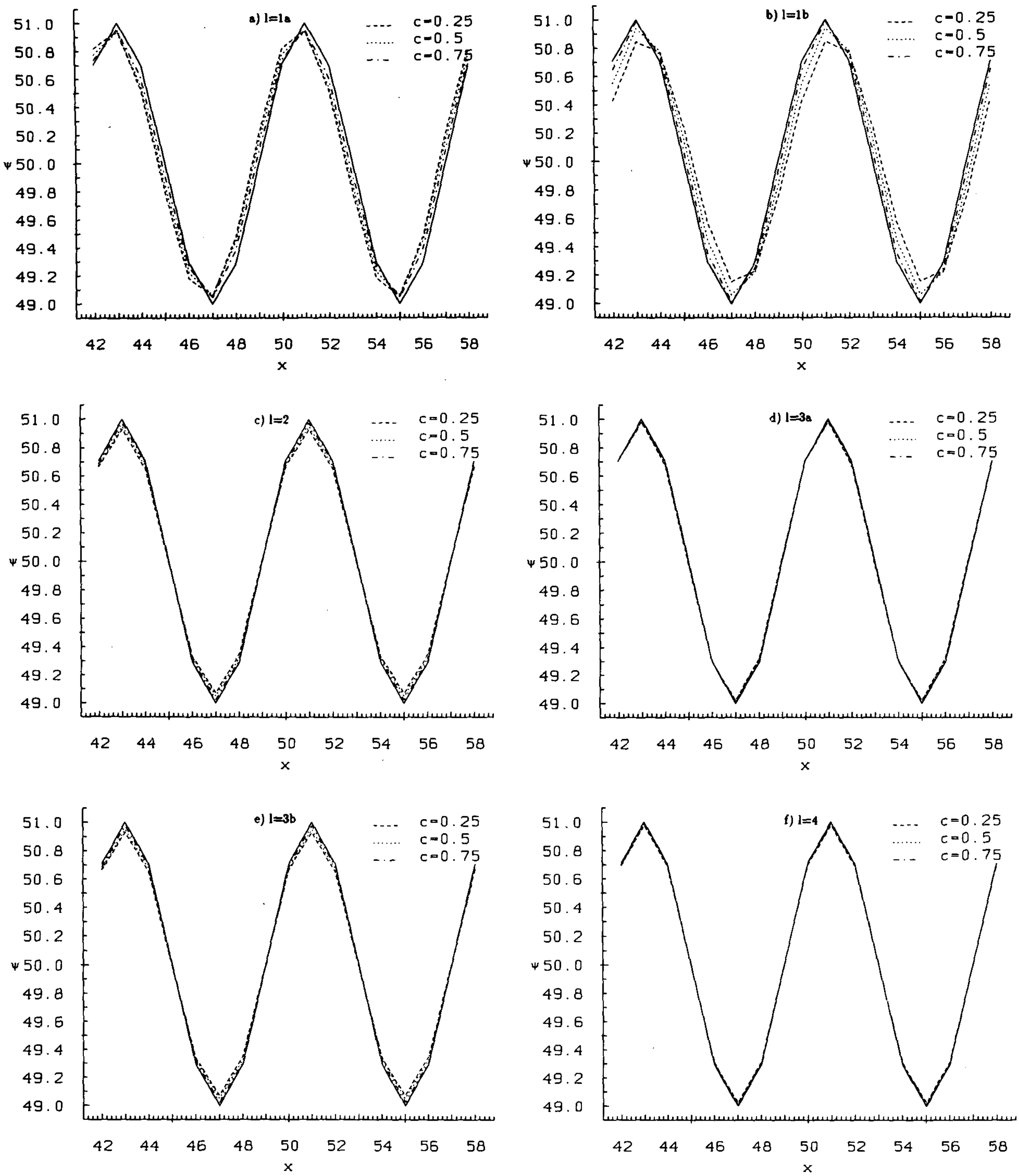

FIG. 2. As in Fig. 1, but for wavelength $8\Delta x$.

in Fig. 3b–f after six full rotations, i.e., 3768 iterations. Table 2 summarizes the main features of the findings. For $l = 1$ and $l = 3$ model versions 1a and 3a of Table 1 have been chosen to fit $\psi_{j,l}(x')$ in (5).

As expected the upstream version ($l = 0$) produces very strong numerical diffusion. After six rotations the maximum of ψ has decreased to only 7% of its initial value, (see Table 2). For $l > 0$ a considerable improvement of the results can be observed with $\psi^n_{\max}$ finally reaching 75–86% of $\psi^0_{\max}$. Model version 1a yields the

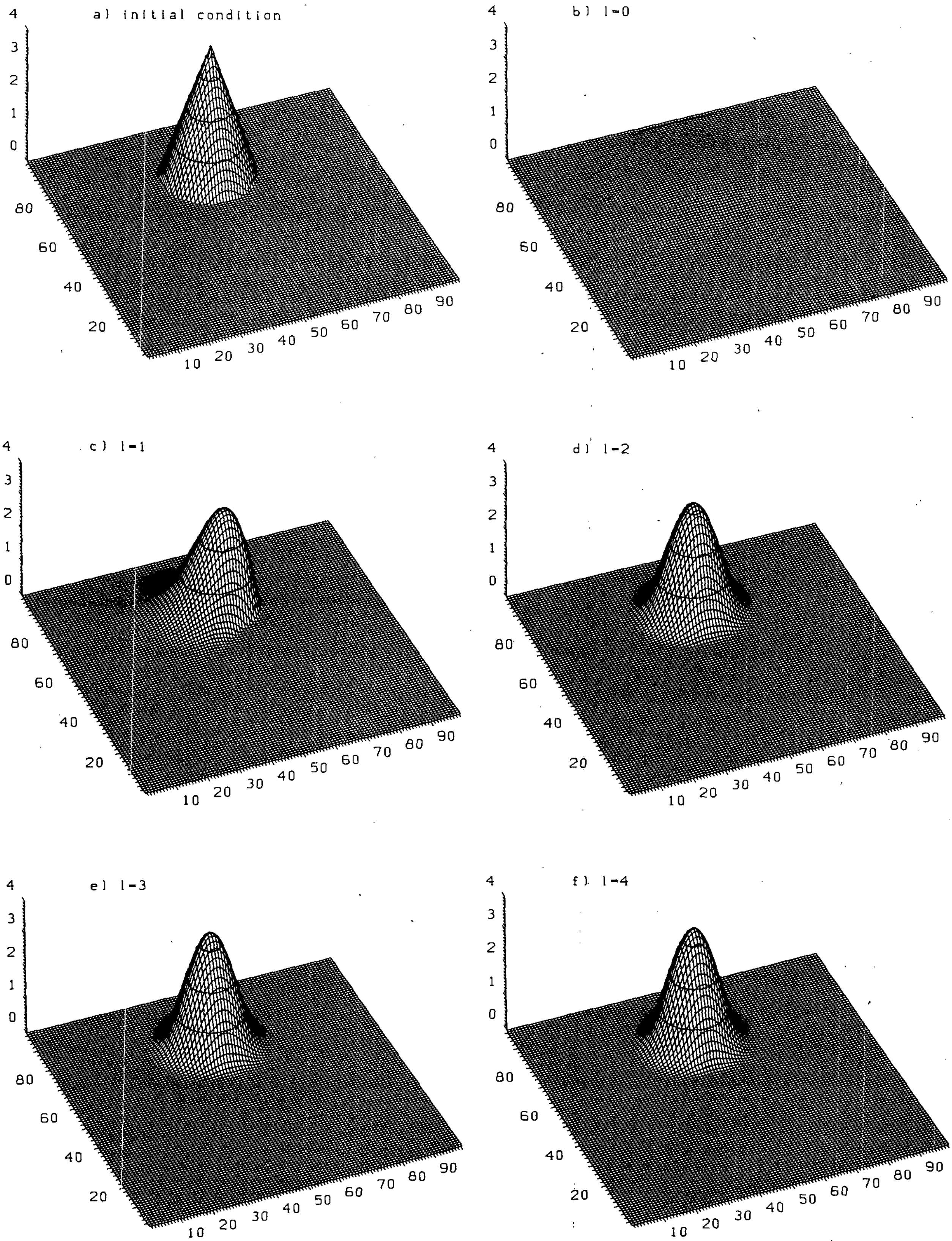

FIG. 3. Results from the rotational flow field experiment after six full rotations for different orders l of polynomials.

TABLE 2. Results from the rotational flow field experiment after 3768 time steps.

l	$\frac{\psi^{3768}_{max}}{\psi^{0}_{max}}$	$\frac{\Sigma (\psi^{3768})^2}{\Sigma (\psi^{0})^2}$	CPU
0	0.07	0.048	1.0
1	0.75	0.793	1.4
2	0.82	0.919	2.0
3	0.86	0.966	2.9
4	0.86	0.966	3.9

strongest deformation of the initial distribution due to the relative large phase errors produced in this particular situation. For $l = 3$ and $l = 4$ the figures are nearly identical.

Observing the computational effort for different model versions ($l = 0, \cdots, 4$) shows, as expected, an increase of the CPU-time required for larger values of l (see Table 2). As already pointed out the version with $l = 2$ may, therefore, be an acceptable compromise between accuracy and numerical efficiency of the scheme. All calculations presented have been performed on a non-vectorizing BULL DPS8/70 computer, even though the algorithm is completely vectorizable coded.

c. Deformational flow field test

In the second experiment the same cone as in Fig. 1a is located in the center of the domain. The velocity field is now modeled by the streamfunction

$$\chi(x, y) = 8 \sin\left(\frac{\pi x}{25}\right) \cos\left(\frac{\pi y}{25}\right) \quad (14)$$

with

$$\begin{aligned} u(x, y) &= -\frac{\partial \chi}{\partial y} \\ &= \frac{8\pi}{25} \sin\left(\frac{\pi x}{25}\right) \sin\left(\frac{\pi y}{25}\right) \\ v(x, y) &= \frac{\partial \chi}{\partial x} \\ &= \frac{8\pi}{25} \cos\left(\frac{\pi x}{25}\right) \cos\left(\frac{\pi y}{25}\right). \end{aligned} \quad (15)$$

The time increment Δt is chosen to 0.7. Obviously, the given χ-distribution as illustrated in Fig 4 yields a strong deformational flow field, which is not typical for atmospheric situations, but serves as a stringent test for the numerical stability of the scheme. As already mentioned by Smolarkiewicz (1982) it is expected that after a long integration time the initial ψ-field is subdivided into two pieces rotating within the areas of the two central vortices of Fig. 4. More recently, Staniforth et al. (1987) presented the analytical solution of this advection problem. They pointed out that due to spatial resolution problems a numerical scheme is not able to reproduce the exact solution after a long integration time. Instead, results should be compared after short integration times, i.e., when the space scales of the analytical solution are still resolvable by the numerical grid mesh.

Thus, Figs. 5a–d depict the numerical solutions obtained with model version 4 after 19, 38, 57 and 75 iterations. These distributions nearly correspond to the analytical solutions presented in Figs. 3a–d of Staniforth et al. (1987). Since within the analytical distribution spatial gradients increase as integration time evolves best results of the proposed scheme are expected at the beginning of the advection process. After 57 time steps the numerical distribution still agrees reasonably well with the analytical result, although the gradients developing at the boundaries of each vortex are not as strong as in the exact solution. (Note in this context the different grid increments of $\Delta x = 1/8$ and 1 in Fig. 3 of Staniforth et al. (1987) and Fig. 5 of this paper, respectively.) After 75 iterations these gradients have nearly completely disappeared at the upper boundaries of the central vortices (see Fig. 5d). The distribution, however, still reproduces the main features of the analytical solution reasonably well. This fact is better illustrated in Fig. 6 depicting a contour plot of Fig. 5d (see also the corresponding Fig. 4a of Staniforth et al. 1987). It can be clearly seen that the solution behaves strong symmetrical as it should. This holds for the whole advection process. Furthermore, the ψ-distribution which is moving towards the center of a vortex is sufficiently well positioned. Figure 7 presents the distribution after 3768 iterations reflecting that the main content of the initial distribution is still located within

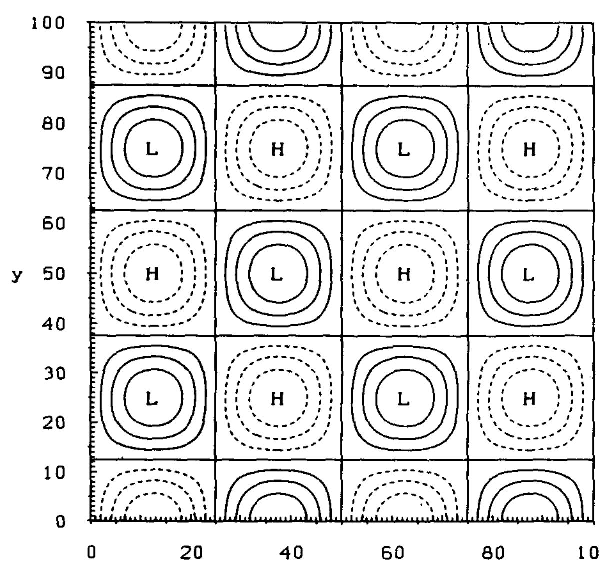

FIG. 4. Streamfunction of the deformational flow field.

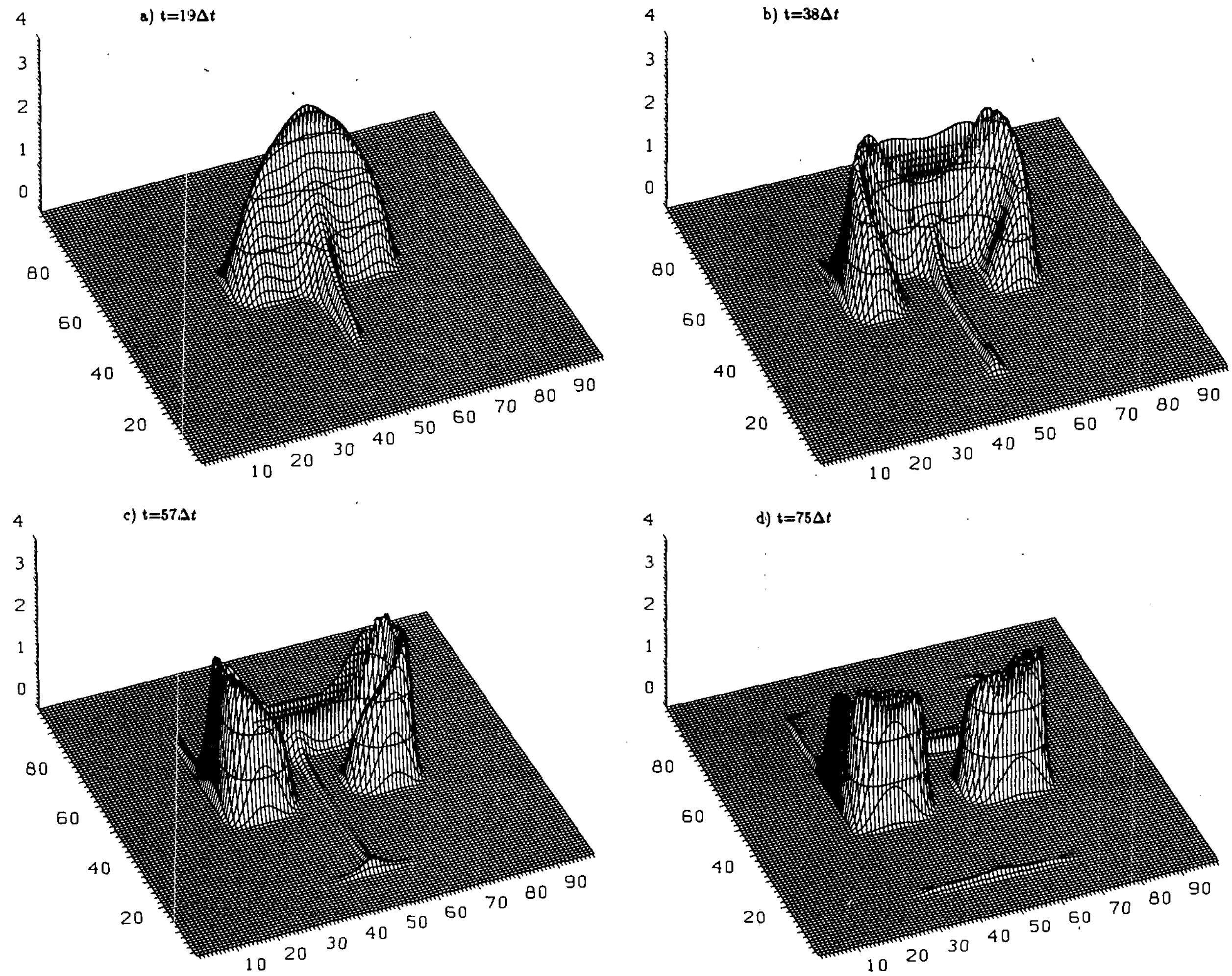

FIG. 5. Results from the deformational flow field test after 19, 38, 57 and 75 iterations.

the central vortices. After this long integration time the maximum of the ψ-field is given by $\psi_{\max}^{3768} = 1.9\psi_{\max}^{0}$ indicating that the scheme is numerically slightly unstable in this particular flow. Due to the weakness of the instability it can be expected that in most atmospheric situations the proposed advection scheme behaves numerical stable since such strong deformational flow fields as above do not exist over longer time scales.

4. Conclusions

Utilizing the integrated flux form of Tremback et al. (1987) a new conservative and positive definite advection scheme has been proposed which is computationally very efficient. The procedure consists in the normalization and limitation of the advective fluxes by upper and lower values. The resulting advection equation is solved by means of the usual upstream procedure. The treatment is formulated in such a manner that it may be applied to other advection schemes which are also written in flux form.

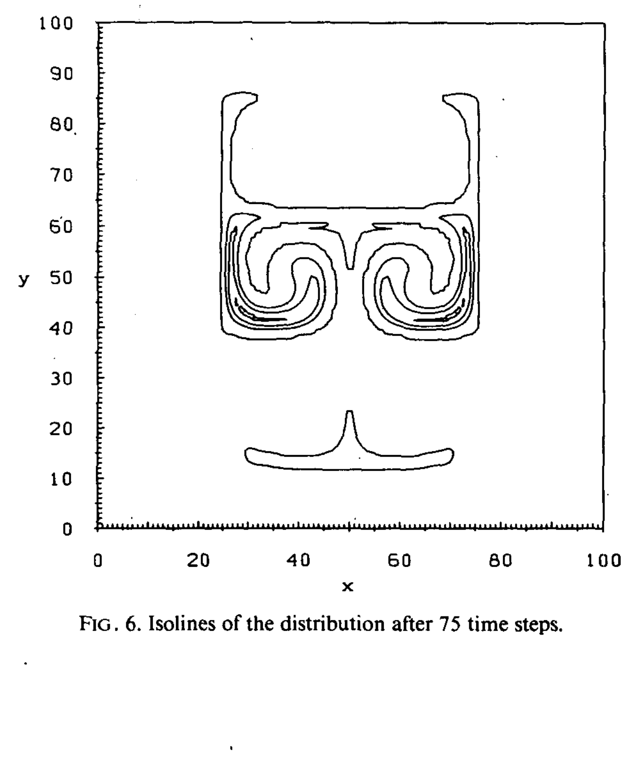

FIG. 6. Isolines of the distribution after 75 time steps.

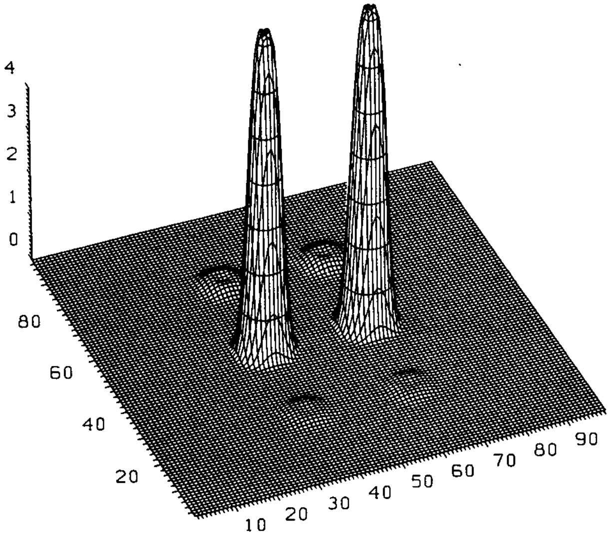

FIG. 7. As in Fig. 5, but after 3768 iterations.

One-dimensional stability tests which have been performed with single Fourier modes of wavelengths $4\Delta x$ and $8\Delta x$ illustrated that higher ordered model versions ($l \geq 2$) yield relatively low phase and amplitude errors. Two-dimensional experiments with a rotating cone elucidated that numerical diffusion decreases considerably when polynomials of order $l \geq 2$ are taken to fit the distribution of the transported quantity. In strong deformational flow fields model version 4 reproduced the analytical solution reasonably well as long as the space scales of the exact solution were resolvable by the numerical grid. The long term solution produced only slight numerical instabilities.

By comparing the numerical efficiency and accuracy of the scheme polynomials of order $l = 2$ turn out to be an appropriate compromise for most atmospheric situations. However, in cases with strong spatial gradients of ψ the order of the polynomials should be increased to obtain more acceptable results.

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