Boundary conditions: lateral, upper, lower

Mikhail Tolstykh
Institute of Numerical Mathematics
Russian Academy of Sciences
and
Hydrometcentre of Russia

Outline

- Generalities
- Lateral boundary conditions for limited area models
- Bottom boundary conditions
- Top boundary conditions

Generalities

- In limited area atmospheric models, only lower boundary is physical, others are arbitrary to a certain extent

- For well-posed problems, one should have one boundary condition per variable per derivative per dimension.

Stability of numerical schemes in presence of boundaries -1

- Only linear, constant coefficient case can be studied.
- Cauchy problem (unbounded or periodic domain) – von Neumann criterion: A numerical method is stable if it does not amplify any of the resolved modes.

$$e^{i(kj\Delta x-\omega n\Delta t)}=\kappa^j z^n$$

(*j* – grid point index in space, *n* – in time). I.e.,$|z|\le 1$ for all $|\kappa|=1$.

- Half-bounded domain ($x\ge 0$) – Godunov-Ryabenki necessary condition: $|z|\le 1$ for all $|\kappa|\le 1$.
- This is not sufficient condition.

Stability of numerical schemes in presence of boundaries -2

Gustaffson, Kreiss, Sundstrom (1972) (GKS-stability):

1) The interior difference formula is stable for Cauchy problem.

2) The model (including discrete boundary conditions) admit no eigenvalues that amplify with each timestep by a constant factor z, $|z|>1$ (I.e. Godunov-Ryabenki condition).

3) The model (including boundary conditions) admit no unforced waves with group velocities directed inward through the boundaries of the domain.

Unstable boundary condition (from D.Durran 1999)

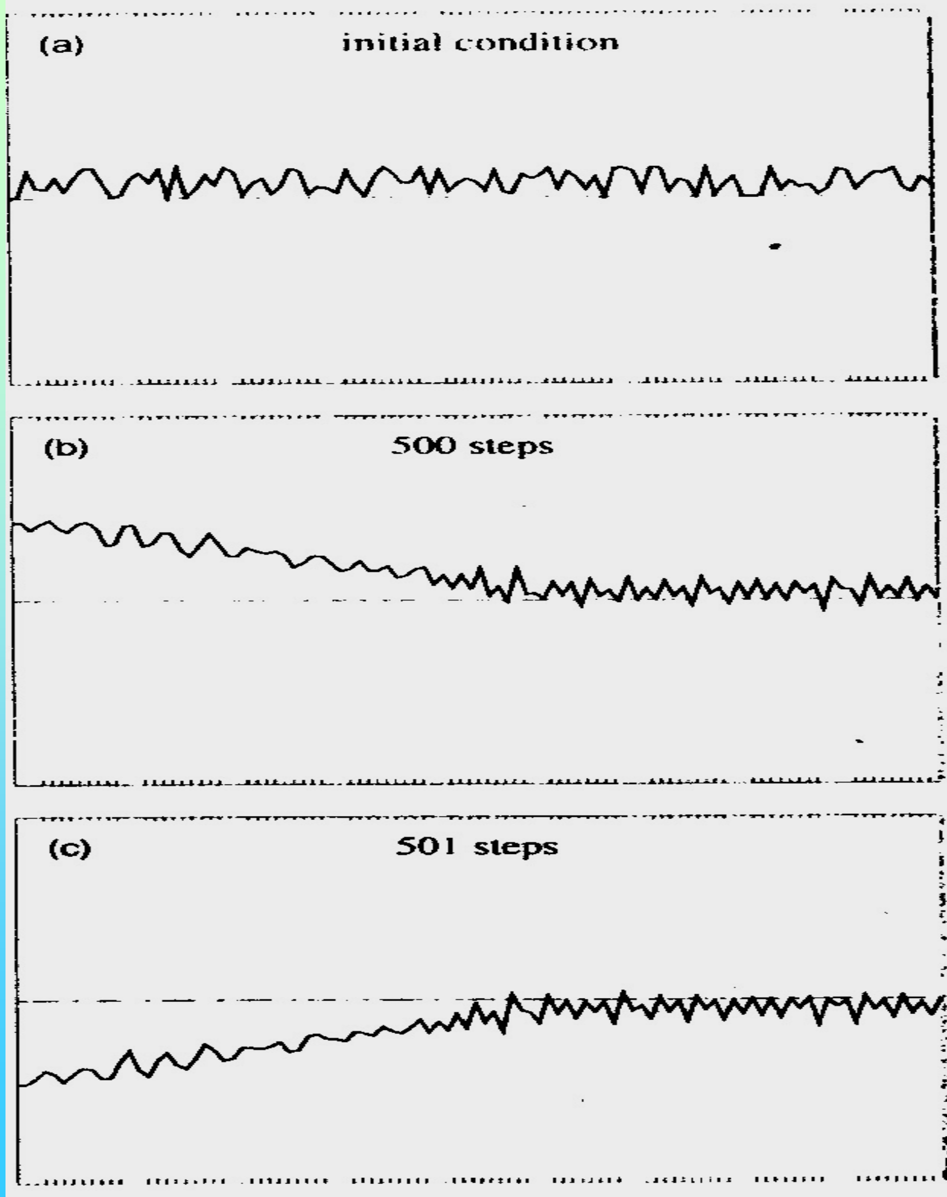
$$\frac{\phi_i^{n+1}-\phi_i^{n-1}}{2\Delta t}+U\frac{\phi_{i+1}^{n}-\phi_{i-1}^{n}}{2\Delta x}=0$$

$$0 \le x \le \infty; U < 0$$

At the outflow boundary

$\phi_0^n = \phi_1^n$ - unstable

$\phi_0^n = \phi_1^{n-1}$ - stable



Why it is unstable?

- Dispersion relation for interior scheme

$$\sin\omega\Delta t = U\Delta t/\Delta x \sin k\Delta x$$

And free wave $e^{i(kj\Delta x - \omega n\Delta t)}$

at the boundary is $1 = e^{ik\Delta x}$ if $\phi_0^n = \phi_1^n$

There is one mode $(\omega, k) = (\pi/\Delta t, 0)$ whose group velocity

$$\frac{\partial\omega}{\partial k} = U\frac{\cos k\Delta x}{\cos\omega\Delta t}$$

In this case it is *–U* so this mode can propagate inward through the downstream boundary

Lateral boundary conditions

- If two boundary conditions are specified per each horizontal dimension, the problem is ill-posed (or overspecified). Let us illustrate this with simple 1D advection equation

Advection equation

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = 0$$

with velocity *U=const* in the interval $0 \le x \le L$ has the exact solution

$$\phi(x,t) = \phi(x - Ut, 0)$$

– one boundary condition required (at the left end if *U>0*).

If one poses second one (or pose it on the other side), the problem is ill-posed. The continuous equation has no (unique, smoothly dependent on boundary condition) solution.

Example (by P. Termonia)

- Leapfrog scheme

$$\frac{\phi_i^{n+1} - \phi_i^{n-1}}{2\Delta t} + U\frac{\phi_{i+1}^{n} - \phi_{i-1}^{n}}{2\Delta x} = 0$$

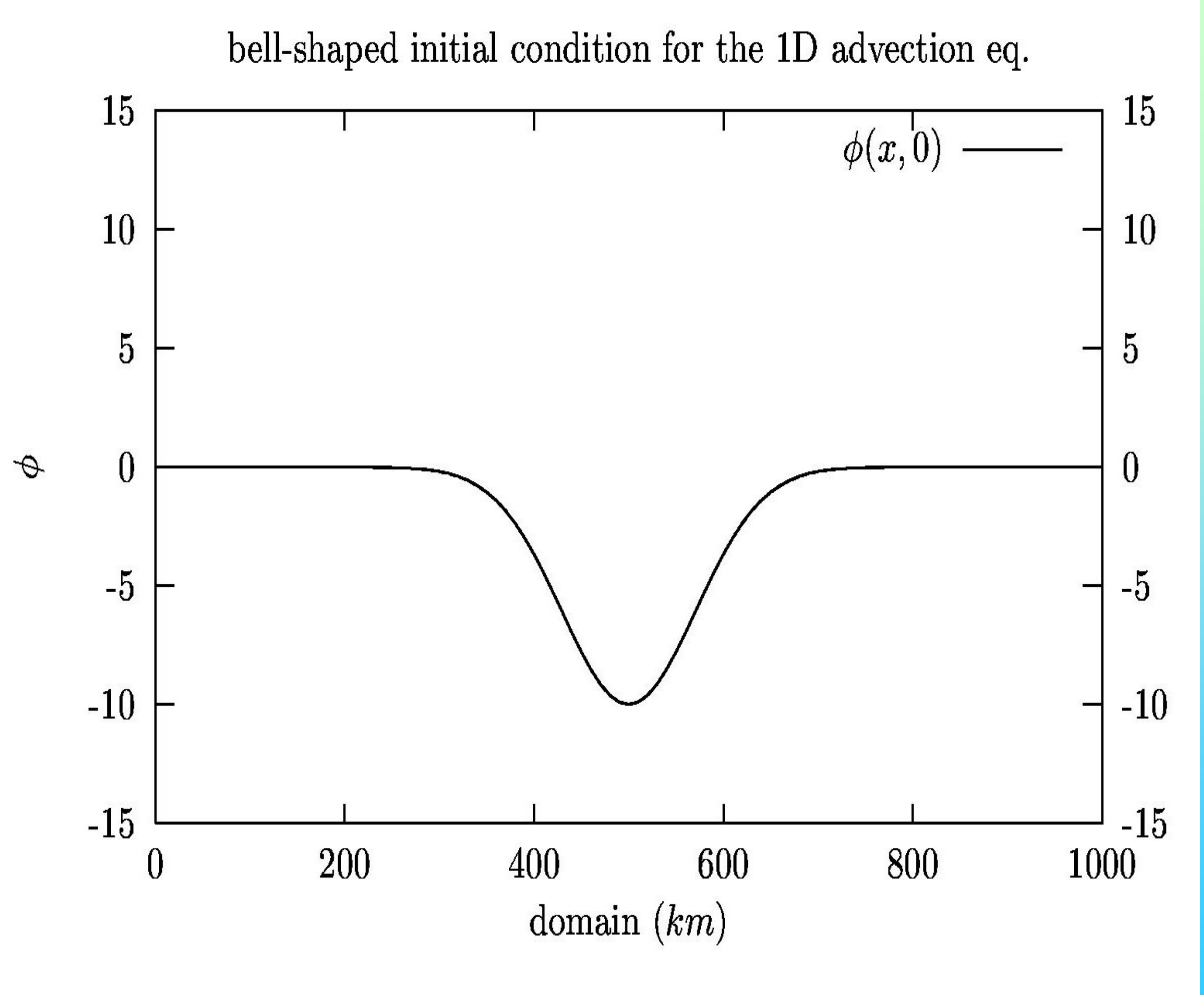
- U=20 m/s, Δx=10 km, Δt=50 s, L=1000 km
- Initial condition

$$\phi(x,0) = A\Phi(x) = Ae^{-100\frac{x-L/2}{L}}$$

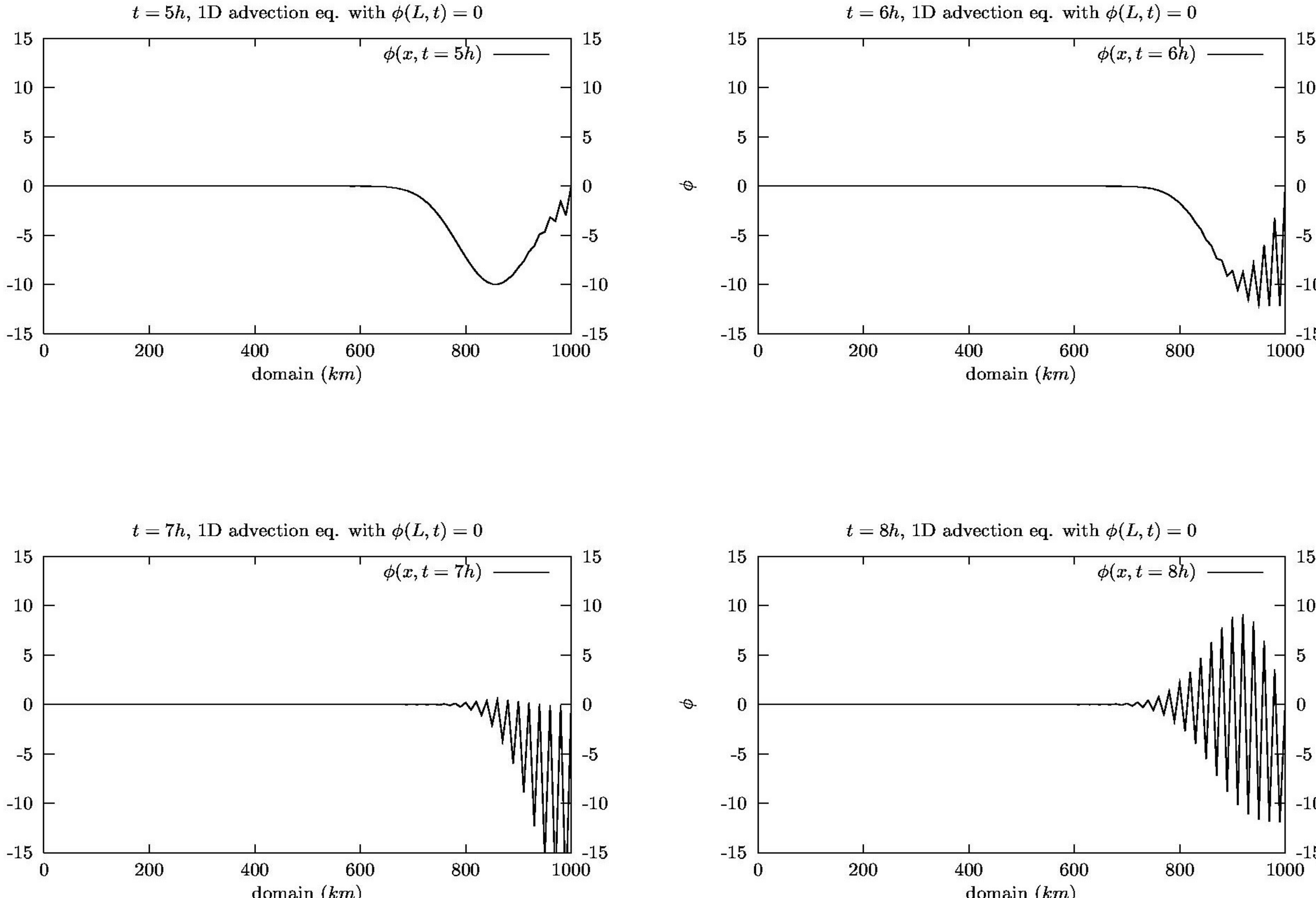
with A=-10.

- Boundary conditions at both ends of the interval =0.

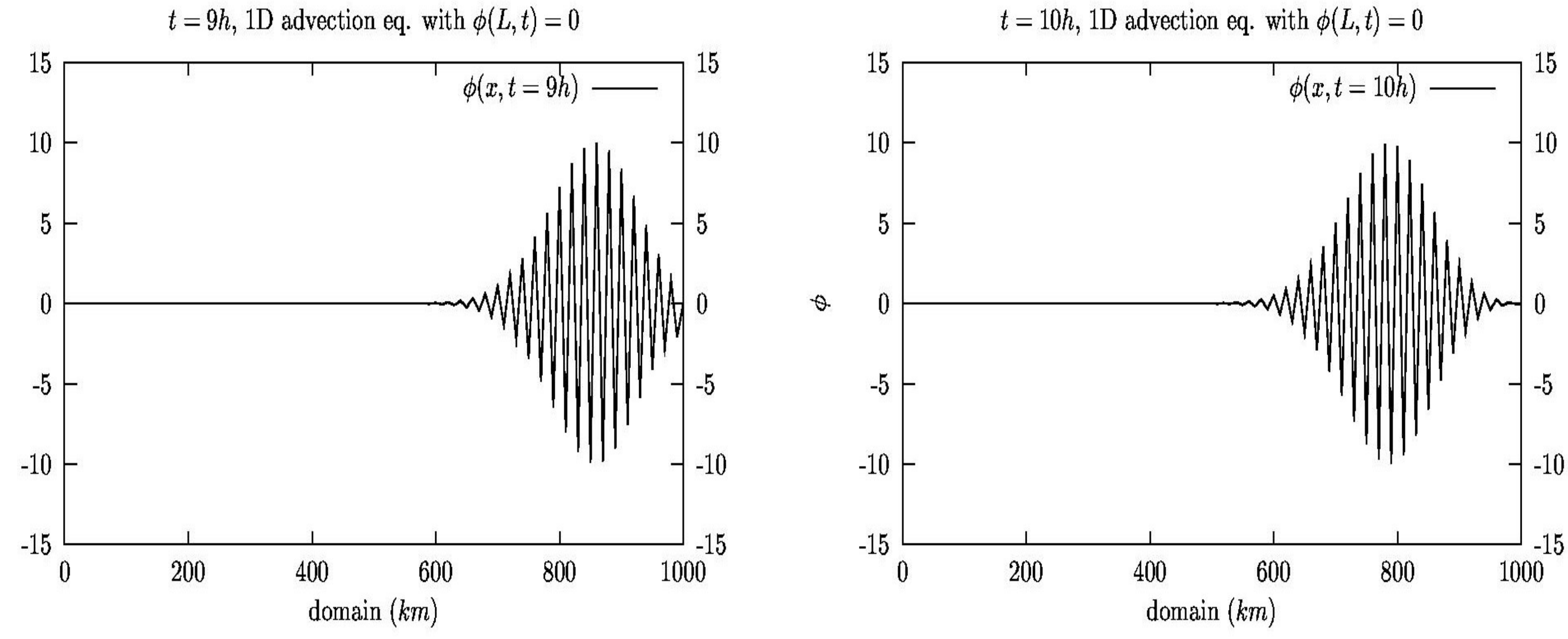
Initial distribution



Example (by P.Termonia) –(continued)



We see spurious wave reflections.



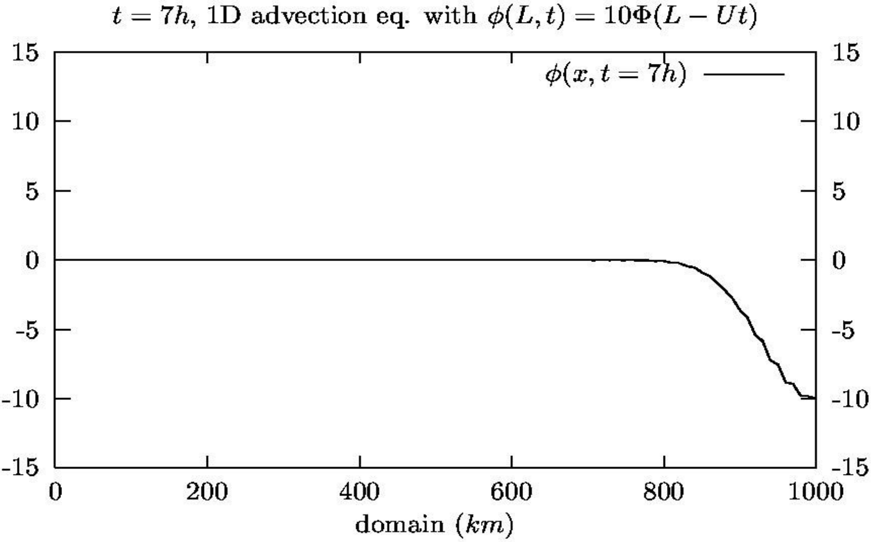
Trying to use well-posed lateral boundary conditions

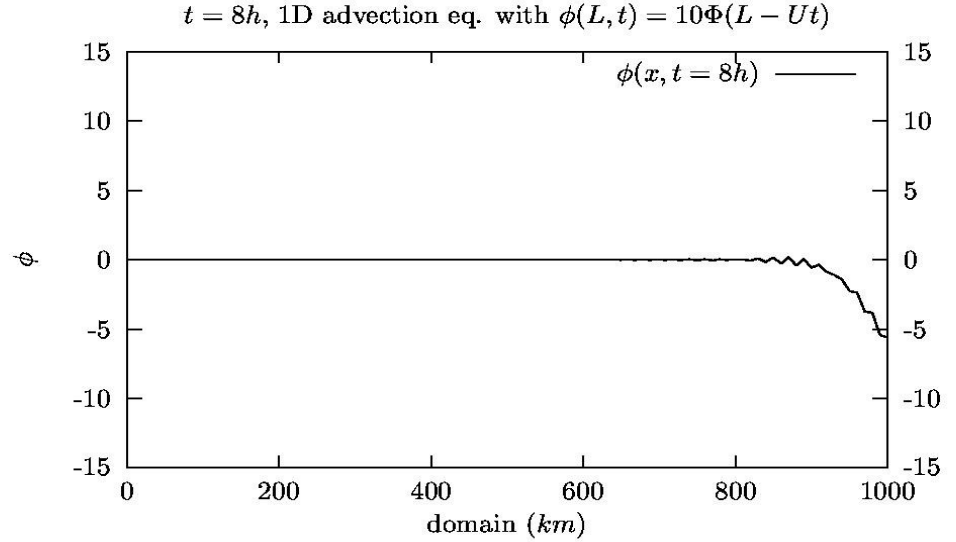
- We know exact solution, let us use it as a boundary condition at the right end. (For a limited area atmospheric model, this could be solution provided by some larger scale model)

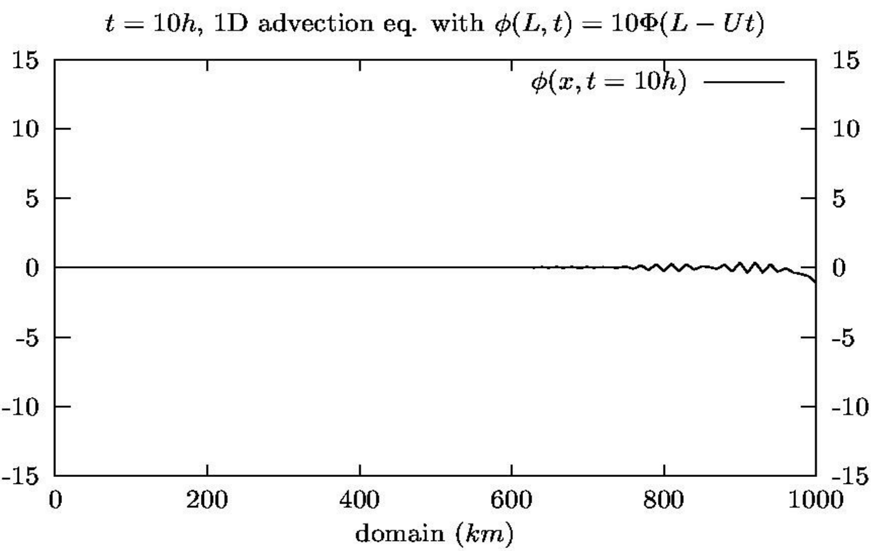
$$\phi(L,t) = A\Phi(L-Ut)$$

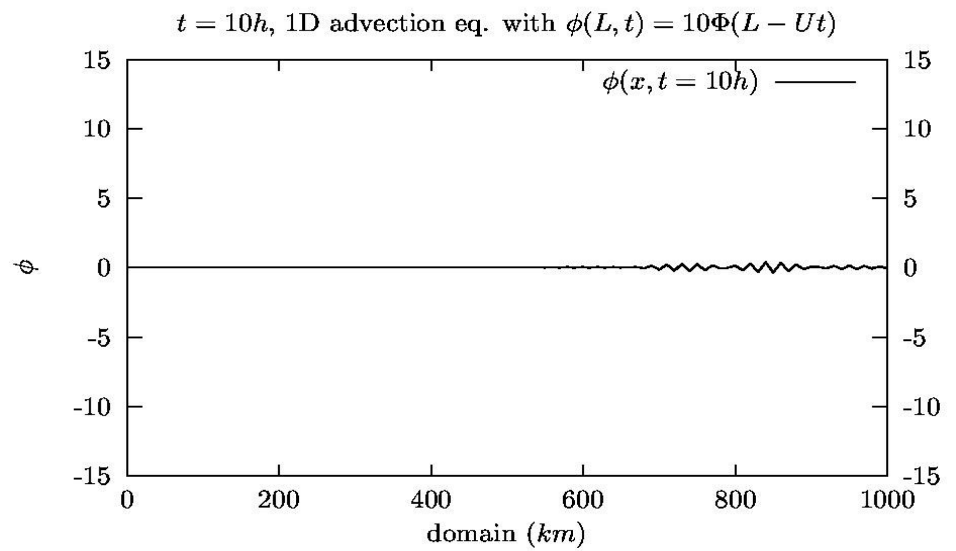
- We see that even in this case there are some reflections, since the solution of approximated problem differs from the analytical one.

Analytical solution as a boundary condition









Lateral boundary conditions (continued)

- It is impossible to impose well posed lateral boundary conditions for numerical scheme having its truncation error with an analytical solution.
- However, it is possible to devise quasitransparent lateral boundary conditions for some solutions of the full atmospheric equations (McDonald, 2005) – considered later.
- Second approach (most used)– simply to filter out spurious wave reflections (Davies scheme).

Flow relaxation (Davies' scheme)

- Modified equation

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = -K(x)(\phi - \tilde{\phi})$$

$\tilde{\phi}$ - is some estimate of solution

K(x) - is non-zero and positive in a boundary zone.

If $\tilde{\phi}$ is (close to) solution of advection equation ,

$$\frac{\partial \phi'}{\partial t} + U \frac{\partial \phi'}{\partial x} = -K(x)(\phi'); \quad \phi' = \phi - \tilde{\phi}$$

$$\phi' = \Phi(x - Ut) e^{-1/U \int_{x_0}^{x} K(y) dy}$$

Flow relaxation -2

- Exponential relaxation near boundary with relaxation e-folding distance *U*
- It is stable provided that the relaxation term is discretized at (*n+1*) time level, as

$$-K_i(\phi_i^{n+1} - \tilde{\phi}_i^{n+1})$$

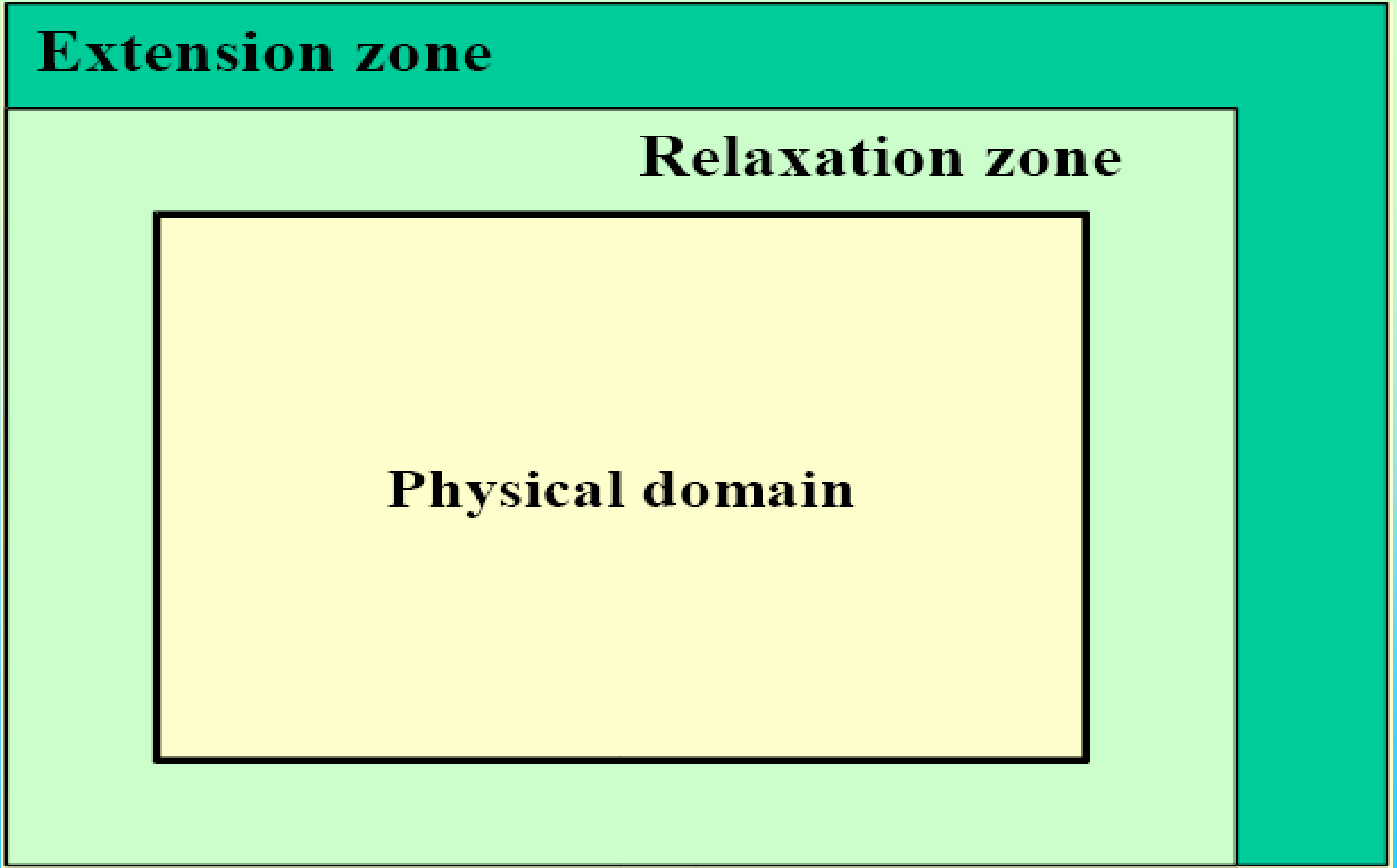
- The solution can be written as a weighted combination of solution to advection equation and boundary condition with the weight of boundary condition varying between 1 at the boundary and 0 inside the domain

Flow relaxation -3

- Some tests have shown that the relaxation distance should be at least 4-5 (8) grid steps (Davies 1983) but not too large as well

- The choice of relaxation functions α that vary between 0 at the end of relaxation zone and 1 at the boundary can be viewed as optimization problem (Lehmann 1993)

Computational domain of ALADIN model



Flow relaxation -4

In ALADIN spectral SISL model, coupling with boundary (large-scale) field is done as follows (Radnoti, MWR 1995) :

1) At the beginning of the time step

$$\Psi \;\; = (1-\alpha)\Psi^{C} + \alpha\Psi^{LS}$$

2) After grid-point computations:

$$(1-\Delta t\,\mathrm{L})\Psi_{\mathrm{t+\Delta t}} = (1-\alpha)\left[\Psi^{\mathrm{exp}}_{\mathrm{t+\Delta t}} + \Delta t\,\mathrm{L}(\Psi_{t-\Delta t} - 2\Psi_{t})\right]$$
$$+\alpha(1-\Delta t\,\mathrm{L})\Psi^{LS}$$

3) There is no grid point calculations in the extension zone

Transparent lateral boundary conditions -1

- Development by A.McDonald after Engquist and Majda1977.
- Trying to prevent fast inertia-gravity waves from entering computational domain, at the same time allowing slow waves to enter.
- Boundary conditions are set as a combination of external and internal fields at the boundary using some decomposition.
- Very promising results demonstrated for 2D hydrostatic primitive equations.

Transparent lateral boundary conditions -2

- Can be done only for linearized equations.
- Results obtained for constant-sign velocity.
- Needs the absence of discontinuities at the boundaries at any time of model integration.
- The discrepancy between linearized and full equations can create noise in real applications.

The method should be developed further. Possibly it can be supplemented by some sort of boundary relaxation.

Space interpolation of lateral boundary conditions

- In the spectral model ALADIN, grid-point interpolation of large scale boundary fields is used operationally. There is an ongoing work by Raluca Radu on spectral interpolation.

Time interpolation of LBC

- Normally, lateral boundary conditions are not available for each time step.
- Christmas storm 1999 – demonstrated that update frequency can play a significant role in success of forecast.
- Two options available in ALADIN:

– linear time interpolation;

- second-order spline. It uses 3 time-levels of the coupling fields at each time (one from behind and two from ahead) (D.Dvorak).

Summary on lateral boundary conditions

- Noise-free lateral boundary conditions in a limited area model is a difficult problem.
- Commonly used Davies relaxation scheme does not provide fully transparent conditions and can affect forecast quality negatively.
- There is an ongoing research on implementing quasitransparent LBCs. The results are encouraging, but there are foreseen difficulties in implementation (avoiding discontinuities at the boundaries etc).

Bottom and top boundary conditions

We will restrict ourselves by mass (pressure)-based coordinate systems.

Bottom boundary conditions -1

- There is a material surface at $\Phi = \Phi_s(x, y)$ so the condition is that velocity normal to this surface is equal to zero.
- If one uses staggered grid in vertical (Lorenc grid – vertical velocity is shifted half grid length from all other variables), this is enough for hydrostatic models. (Implicitly, free-slip conditions for *u* and *v* are assumed).

Bottom boundary conditions -2

In an NH model, conditions for some other variables (depending on model design) may be needed at the surface:

- A relationship on vertical acceleration (ALADIN NH) (Benard 2004)
- Free-slip (for horizontal velocity components)
- Extrapolation to (or below) the ground (based on constant gradient) (for pressure departure in LM)
- Diagnostic relationship for ω -velocity at the surface based on free slip condition (NH HIRLAM, LM)
- In the case of extrapolation, take care about GKS stability of numerical scheme.

Top boundary conditions -1

- In general, there is no material surface.
- Material surface can be introduced (example for z-system)

$$\frac{dp_T(x,y,t)}{dt} - gw_T(x,y,t) = 0$$

- This leads to severe complications in the algorithm (one more equation to be solved).
- Setting $p_{top} = 0$ describes vertically unbounded atmosphere. This is the case of ALADIN NH. However, this can be undesirable in some applications (jump in vertical resolution or many extra levels).

Top boundary conditions -2

- In practice, vertical velocity = 0 (more precisely, velocity normal to the upper model coordinate surface) at the top in most cases.
- Causes wave reflection similarly to lateral boundaries.
- Free-slip conditions are used for other variables. This means that the vertical derivatives of these variables are equal to zero and there is no mass and heat transfer across the boundary.

Top boundary conditions -3

Radiation boundary condition can be imposed by diagnostic relationship between pressure and vertical velocity at the top (Klemp, Durran 1983; Bougeault 1983). However, they are formulated in terms of vertical wavenumbers and frequencies and are difficult to implement.

Absorbing layer -1

- The approach used to prevent the spurious reflections from the top of the atmosphere.
- Explicit absorbing layer – increasing parameterized horizontal diffusion while approaching the top.
- Implicit absorbing layer – coarsening the vertical resolution while approaching the top (UKMO, LM, …).
- Advantage - it is simple to implement.

Absorbing layer -2

- Disadvantages:

- needs multiple layers (the horizontal diffusion coefficient cannot change too rapidly, otherwise it will create reflections itself).

- needs tuning.

Conclusions on bottom and top boundary conditions

- Bottom boundary conditions is physical one and represent no problem.
- Top boundary condition – the approach generally used is to set the vertical velocity to zero and to have an explicit or implicit absorbing layer.

General conclusion

The problems of boundary conditions in atmospheric limited area models have their solutions but are far from being closed.

Thank you!

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