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Semi-Implicit, Semi-Lagrangean (SISL) Methods

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Introduction

- SISL = Semi-Implicit, semi-Lagrangean
- Introduced by Robert 1981
- First NWP models 1985 – 87, main-stream development 88-93
- ECMWF, HIRLAM, Meteo-France: implementation in operational WP in 1995, Canadian MC2 - 1997
- Main advantage: stability (approximately 1km of grid-step corresponds to 1 min in time step) and accuracy

Lagrangian equations

On the example of semi-elastic dynamics in pressure-coordinates

$$\frac{d\omega}{dt} = -\frac{p^2}{H^2}\frac{\partial \phi}{\partial p}$$

$$\frac{d\mathbf{v}}{dt} = -\nabla(\phi + \varphi) - \mathbf{f} \times \mathbf{v}$$

$$c_p \frac{dT}{dt} = RT\frac{\omega}{p}$$

$$\frac{dp_s}{dt} = \omega\Big|_{p_s}$$

$$\nabla \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0$$

$$\frac{d\varphi}{dp} = -\frac{RT}{p}$$

v –horizontal wind vector, ***ω*** – vertical (omega-)velocity, ***H=RT/g*** – scale height, ***Φ*** – non-hydrostatic geopotential, φ - hydrostatic geopotential,

R, C_p – gas constants, ***p*** – pressure, ***p_s*** - surface pressure

General NH pressure-coordinate equations

Isobaric height equation	$\frac{\mathrm{d}z}{\mathrm{d}t} = w \; ;$
Vertical momentum equation	$n \frac{\mathrm{d}w}{\mathrm{d}t} = g(1 - n) \; ;$
Horizontal momentum equations	$n \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = - g\nabla z - n\mathbf{f} \times \mathbf{v} \; ;$
Temperature equation	$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{RT\omega}{c_p p};$
Density equation	$\frac{\mathrm{d}n}{\mathrm{d}t} + n\left(\nabla_p \cdot \mathbf{v} + \frac{\partial \omega}{\partial p}\right) = 0 \; ;$
Surface pressure equation	$\frac{\partial p_s}{\partial t} + \nabla \cdot \int_0^{p_0} \mathbf{v} dp = 0 \; ;$
Metric equation	$\frac{p}{H} \frac{\partial z}{\partial p} = -n \; .$

To get HS PE, it is suficcient to put $n \to 1$ everywhere

Semi-elastic pressure-coordinate model

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = -\frac{1}{p_s}\int_0^1 \nabla\cdot\mathbf{v}\ m\ d\eta - \frac{\mathrm{d}\ln\hat{p}_s}{\mathrm{d}t} \equiv F_\chi, \tag{10a}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{p^2}{mH^2}\frac{\partial\phi}{\partial\eta} + \omega\left(\frac{c_v\omega}{c_p p} - \frac{A_T}{T} - \frac{\mathrm{d}\ln R}{\mathrm{d}t}\right) + A_\omega \equiv F_\omega, \tag{10b}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla_p(\varphi+\phi) - f\mathbf{k}\times\mathbf{v} + \mathbf{A}_\mathbf{v} \ \equiv \mathbf{F}_\mathbf{v}, \tag{10c}$$

$$\frac{\mathrm{d}T'}{\mathrm{d}t} = \left(\frac{RT}{c_p p} - \frac{\partial T^0}{\partial p}\right)\omega - \frac{\partial T^0}{\partial t} + A_T \equiv F_T, \tag{10d}$$

$$\nabla_p\cdot\mathbf{v} + \frac{1}{m}\frac{\partial\omega}{\partial\eta} = 0 \tag{10e}$$

In these equations, $\chi = \ln(p_s/\hat{p}_s)$, omega velocity is $\omega = dp/dt$, ϕ is the nonhydrostatic geopotential perturbation (a pressure-coordinate equivalent of the common z-coordinate nonhydrostatic pressure perturbation; $H = RT/g$ is the scale height.

Short-notation Eq. of motion in Lagrangean presentatio

Lagranean equation of motion

$$\frac{d\psi}{dt} = F(\psi)$$

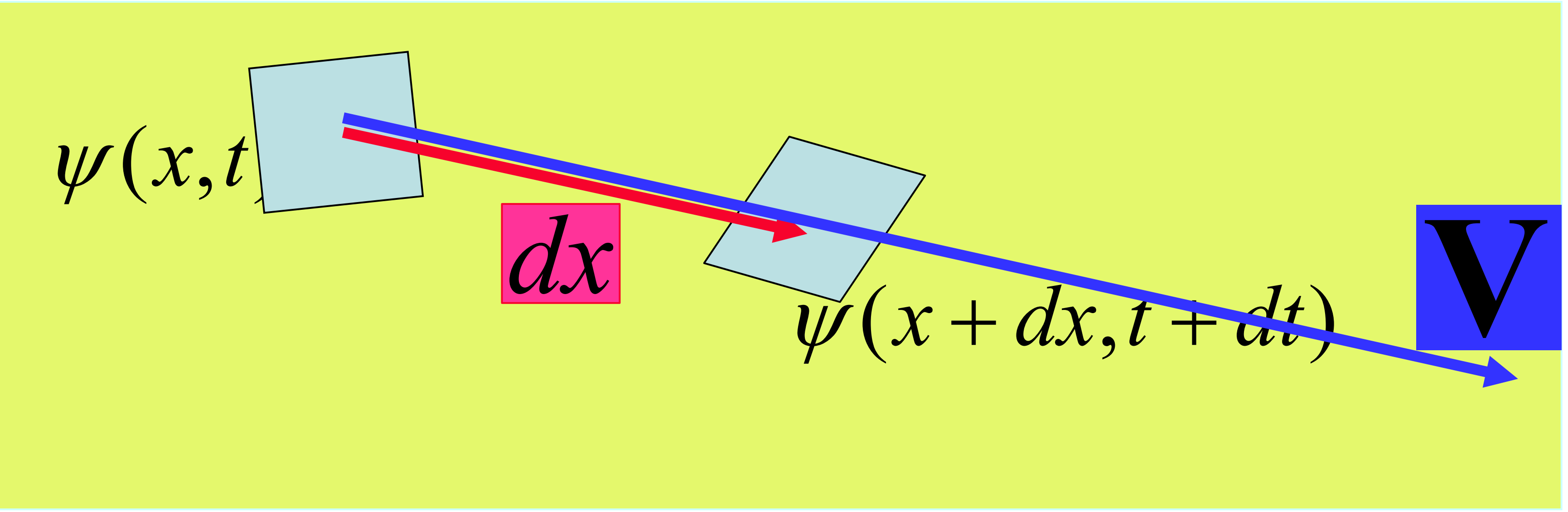
$$\psi = \begin{matrix} u \\ v \\ w \\ T \end{matrix} \qquad F(\psi) = \begin{matrix} F_u \\ F_v \\ F_w \\ F_T \end{matrix}$$

Eulerian equation of motion

$$\frac{\partial \psi}{\partial t} + \mathbf{V} \cdot \nabla \psi = F(\psi)$$

Outlines of SL scheme

$$\frac{d\psi}{dt} = F(\psi)$$

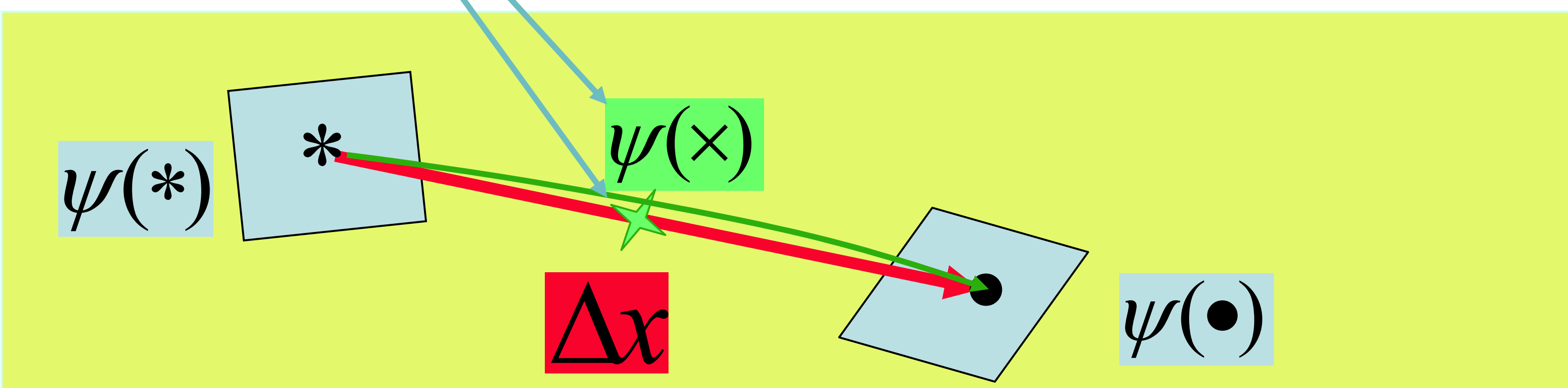


Discretization

$$\frac{d\psi}{dt} = F(\psi)$$

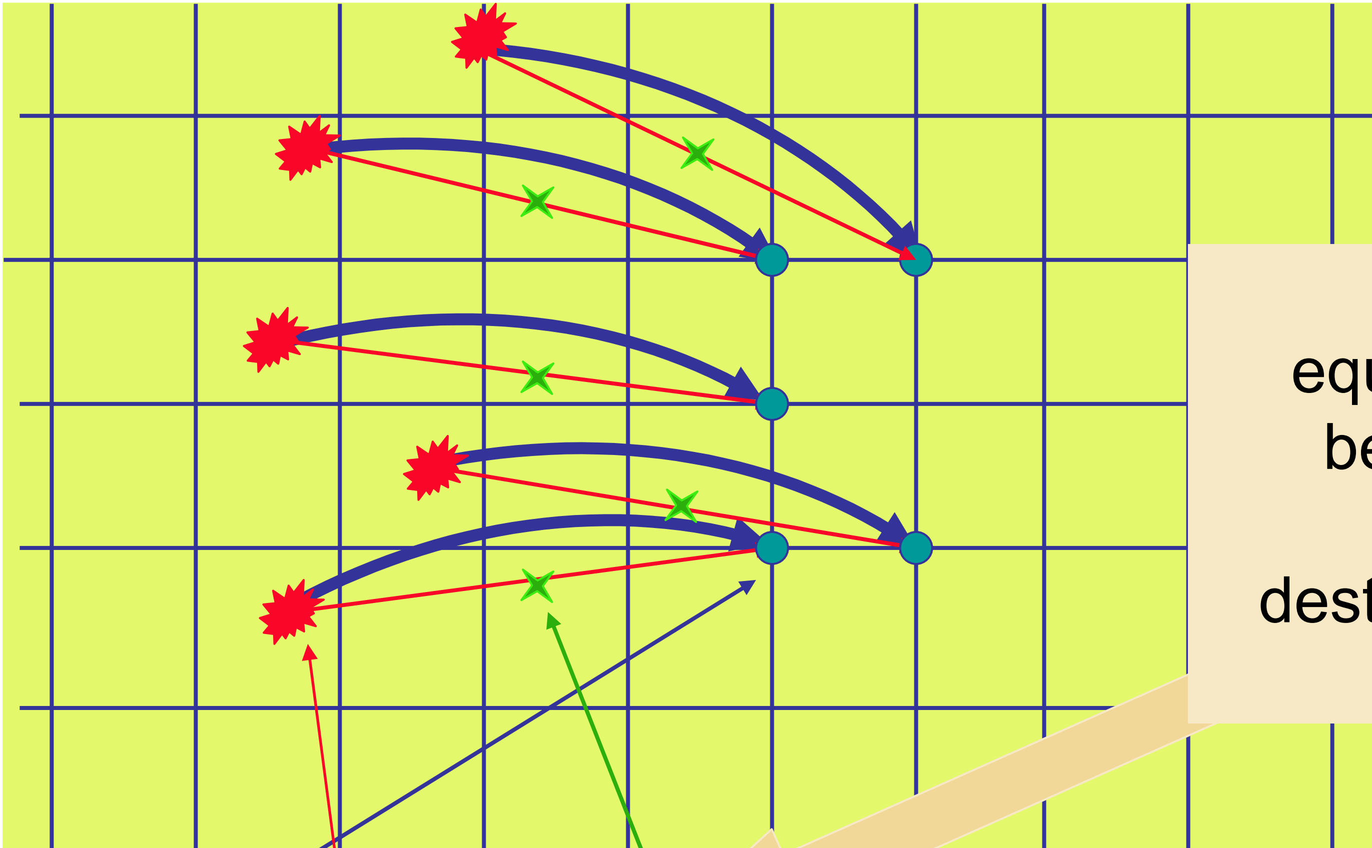
(Explicit case)

$$\frac{\psi(\bullet) - \psi(*)}{\Delta t} = F[\psi(\times)]$$



$$(\bullet) = (\mathbf{x}, t),\ (\times) = (\mathbf{x} - \Delta\mathbf{x}/2, t - \Delta t/2),\ (*) = (\mathbf{x} - \Delta\mathbf{x}, t - \Delta t)$$

SL in action

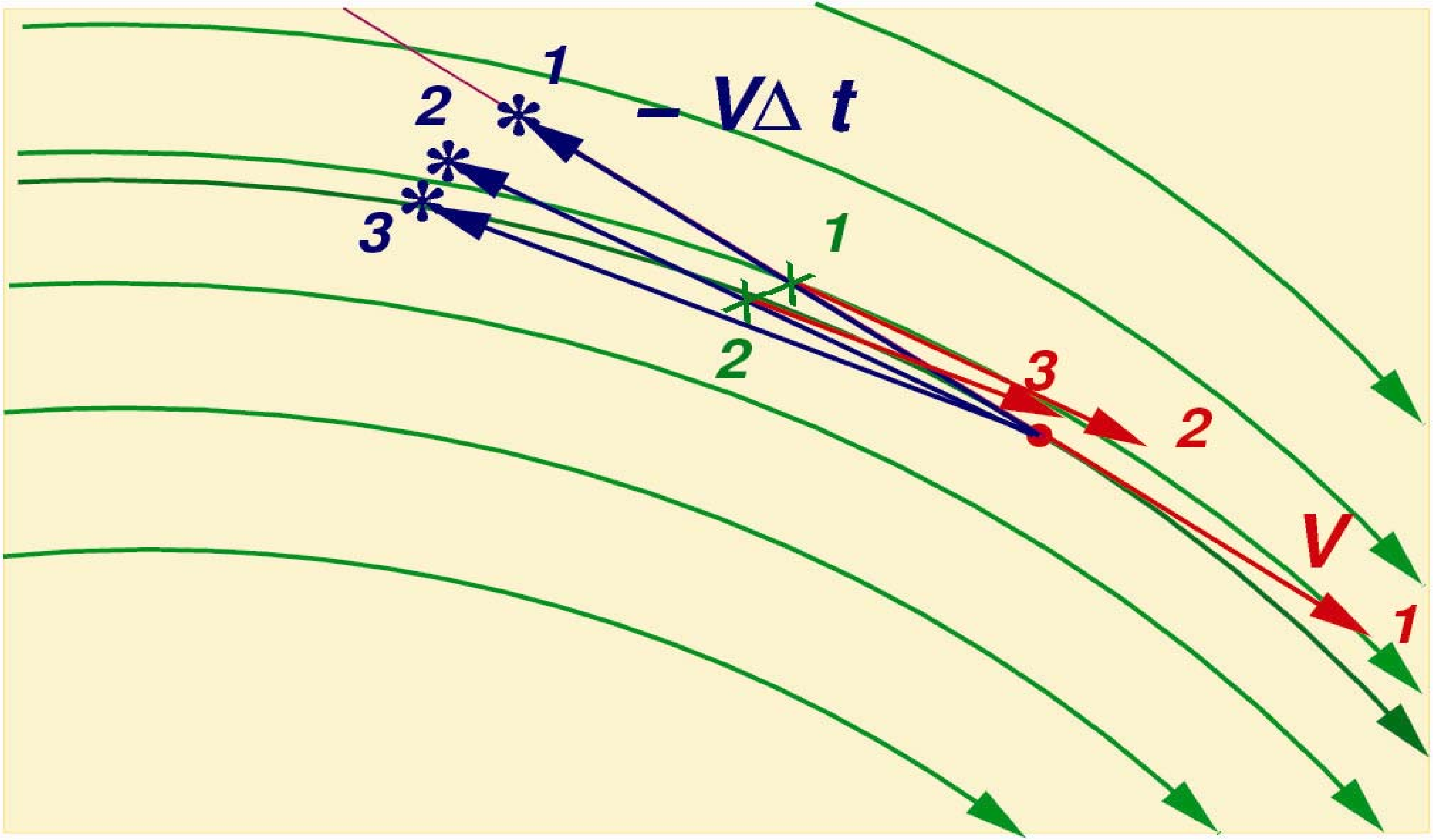


Lagrangean equation has to be solved with respect to destination-point value ***ψ* (●)**

$$\frac{\psi(\bullet)-\psi(\ast)}{\Delta t} = F[\psi(\times)]$$

***ψ* (●)** is the final value on the node, departure value ***ψ* (✸)** and forcing *F[**ψ**(**X**)]* are interpolated

Trajectory calculation



1. Departure point (*) is find after 2-3 iterations

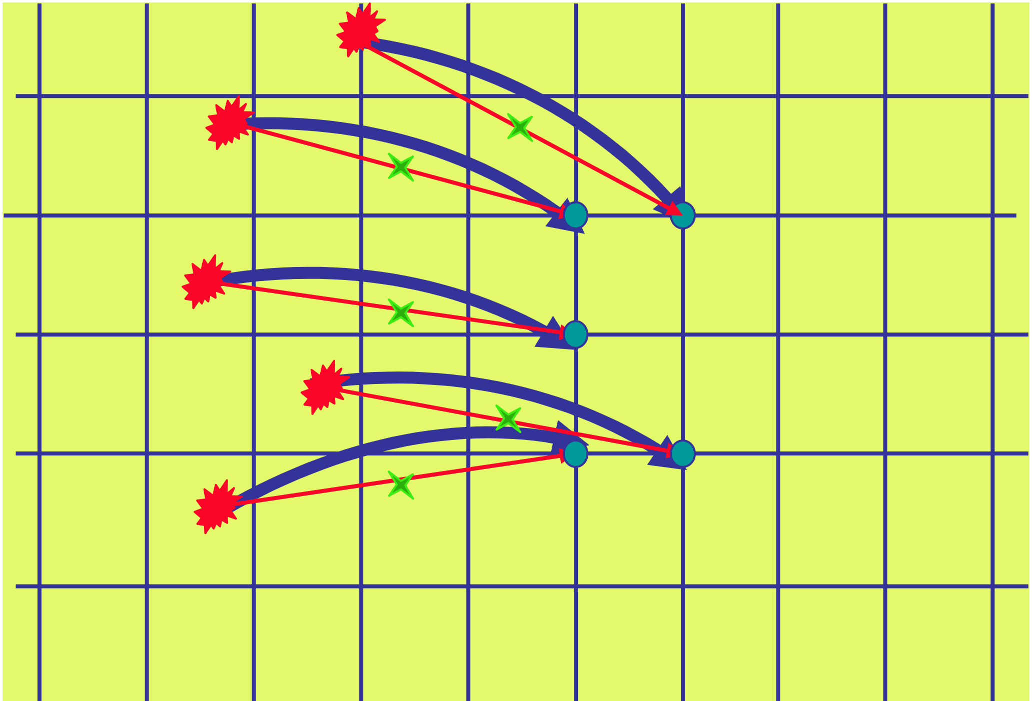
2. Particle trajectories should be locally (during Δt) arcs of circle

As the departure points are located in between nodes, interpolation is required

for departure field ψ (*)

- for intermadiate field ψ (x)

The interpolation routines are the essential components of SL numerics



That was the basic idea how SL scheme works

REFINEMENTS

FOLLOW HEREAFTER

Implicit scheme :

$$\frac{\psi(\bullet)-\psi(*)}{\Delta t}=F[\psi(\times)]$$

$$\frac{\psi(\bullet)-\psi(*)}{\Delta t}=\frac{1}{2}\{F[\psi(\bullet)]+F[\psi(*)]\}$$

Again, one needs to solve Eq. with respect to $\psi\,(\bullet)$. However, this time unknown $\psi\,(\bullet)$ on the right side, also.

Solution is complicated and time consuming. Can be solved iteratively, but many iterations are needed.

Semi-implicit (SI) scheme

$$F(\psi) = \mathbf{L}\psi + \mathbf{N}(\psi)$$

$$\mathbf{L}\psi - \textit{linear part}$$

$$\mathbf{N}(\psi) = F(\psi) - \mathbf{L}\psi$$

$$- \textit{nonlinear} \quad \textit{residual}$$

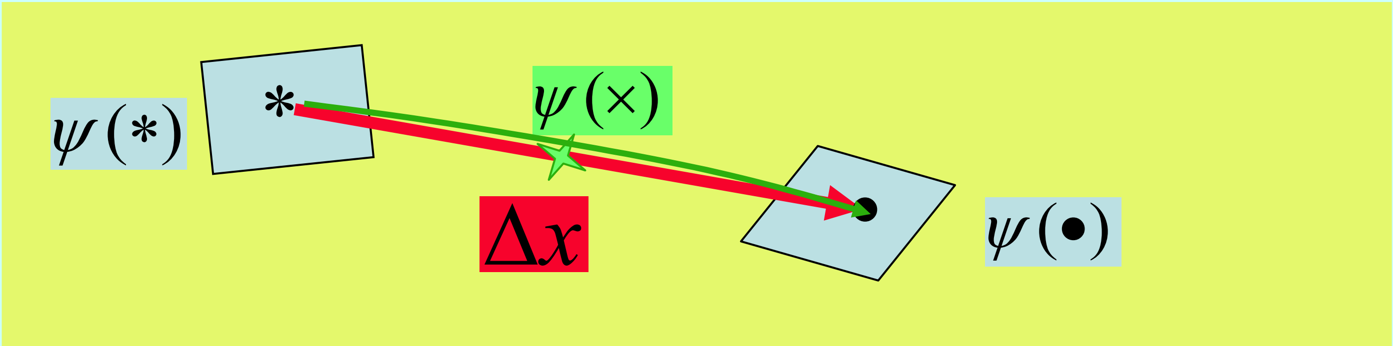
$$\frac{\psi(\bullet)-\psi(*)}{\Delta t}=F\left[\psi(\times)\right]$$

Equivalent presentation

$$\frac{\psi(\bullet)-\psi(*)}{\Delta t}=\mathbf{L}\psi(\times)+\mathbf{N}\left[\psi(\times)\right]$$

Modification

$$\frac{\psi(\bullet)-\psi(*)}{\Delta t}=\frac{1}{2}\left[\mathbf{L}\psi(\bullet)+\mathbf{L}\psi(*)\right]+\mathbf{N}\left[\psi(\times)\right]$$



Solving of

$$\frac{\psi(\bullet)-\psi(*)}{\Delta t}=\frac{1}{2}[\mathbf{L}\psi(\bullet)+\mathbf{L}\psi(*)]+\mathbf{N}[\psi(\times)]$$

With respect to ***ψ (•)*** gives

$$\left[1-\frac{\Delta t}{2}\mathbf{L}\right]\psi(\bullet)=\left[1+\frac{\Delta t}{2}\mathbf{L}\right]\psi(*)+\mathbf{N}[\psi(\times)]$$

THIS IS THE MAIN EQUATION OF SISL scheme

(actually, a similar Eq. exists in any SI scheme)

Elliptic (Laplace, Helmholtz) operator

THE FINEST POINT:

CHOISE of the LINEAR PART

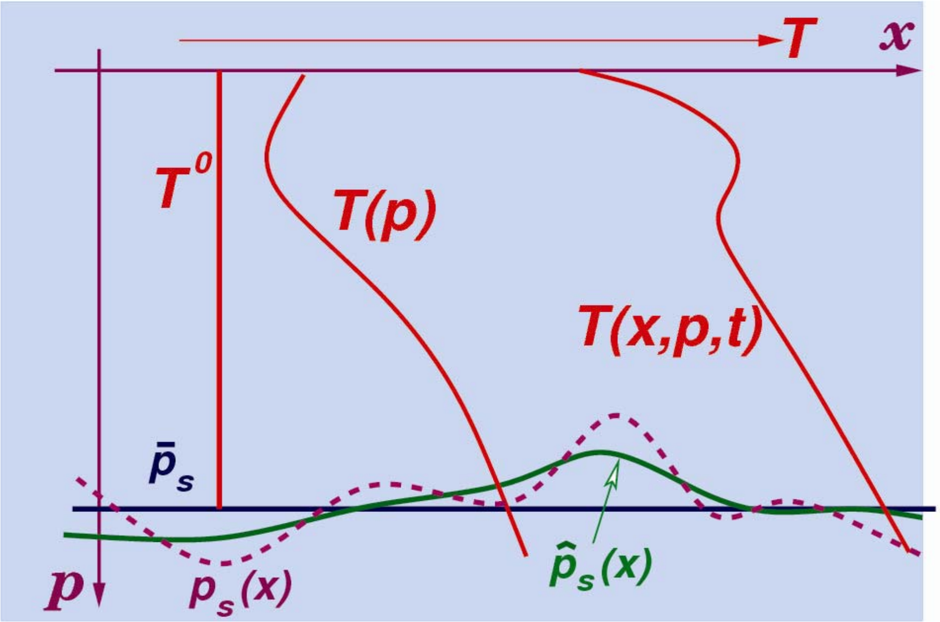
This is the point where various (dynamically identical) SISL models start to differ

The linear part Lψ is actually a part of F(ψ), linearised with respect to the reference state, specified by reference temperature $\boldsymbol{T^{ref}}$ and surface pressure $\boldsymbol{p_s^{ref}}$.

General requirement: Lψ *must handle (fast) buoyancy and acoustic waves*

$$T^{ref} = T(x,p,t), T(p), T^0$$

$$p_s^{ref} = p_s(x,t), \hat{p}_s(x), \bar{p}_s$$



$$T^{ref} = T(x,p,t), T(p), T^0$$

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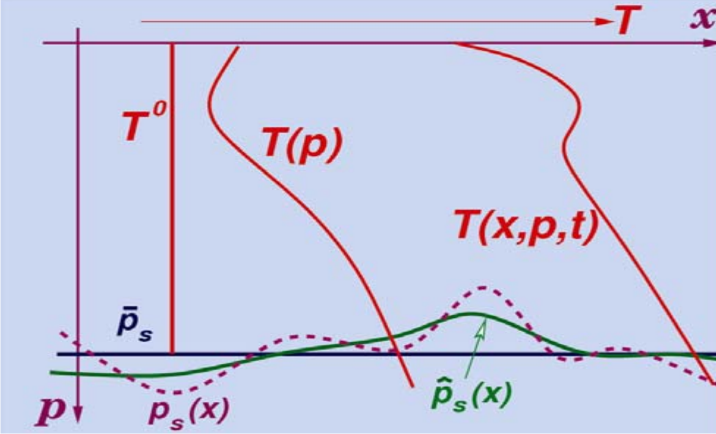


The closer is reference state to the actual stratification and pressure distribution, the smaller is nonlinear residual and the larger is numerical stability.

However: the closer is reference state to the actual stratification and pressure distribution, the more complicated and expensive is solution of main elliptic equation

Thus, reasonable compromise between requirements of numerical simplicity/stability and of the closeness of the reference state to real atmosphere is necessary

ECMWF and HS HIRLAM make use of constant T^{ref} and p_s^{ref} .

$$T^{ref} = T^0, p_s^{ref} = \bar{p}_s$$

Meteo-France' NH model ALADIN makes use of two different constant temperatures, and constant surface pressure.

$$T^{ref} = T^0, T^1,$$

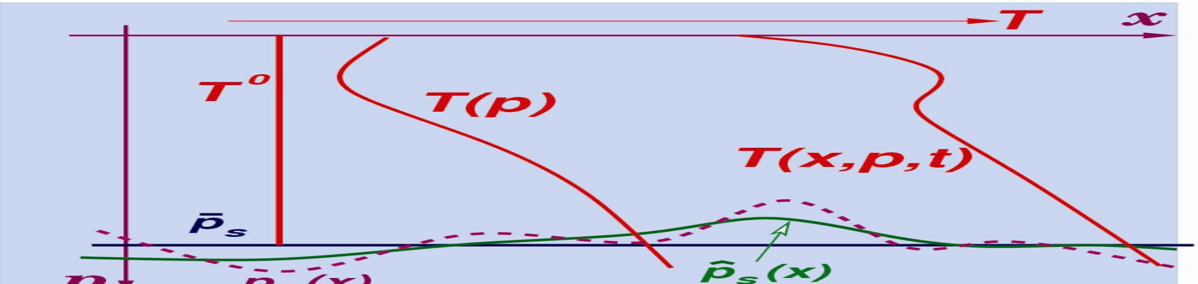
$$p_s^{ref} = \bar{p}_s$$

However, as the full-implicit scheme is solved iteratively, effectively the actual T and p_s are used as the reference fields.

In the NH SISL HIRLAM, horizontal area-mean reference temperature is used and constant reference pressure; both are time-adaptive.

$$T^{ref} = T(p,t),$$

$$p_s^{ref} = \bar{p}_s(t)$$



Choice of reference fields T^{ref} and p_s^{ref} is crutial for stability.

•The linear part is always unconditionally stable (which will be demonstrated further on)

•However, the nonlinear residual $N(\psi)$ can cause numerical instability, if the reference fields T^{ref} and p_s^{ref} are chosen unsuitably

This property is model sensitive: HS primitive equation model (ECMWF, HS HIRLAM), the full elastic model (ALADIN) and semi-elastic model (NH HIRLAM) behave differently with respect to reference field specification.

That is the reason, why T^{ref} and p_s^{ref} are chosen differently in these models.

Stability of SISL

For elementary demonstration of **stability of the linear part of SISL**, the next simple model is instructive. Lagrangean Eq. of motion becomes a scalar complex wave equation:

$$\mathbf{L} = \mathbf{i} / t_0$$

$\mathbf{i}$ – *imaginary unit*,

t_0 – *real const*

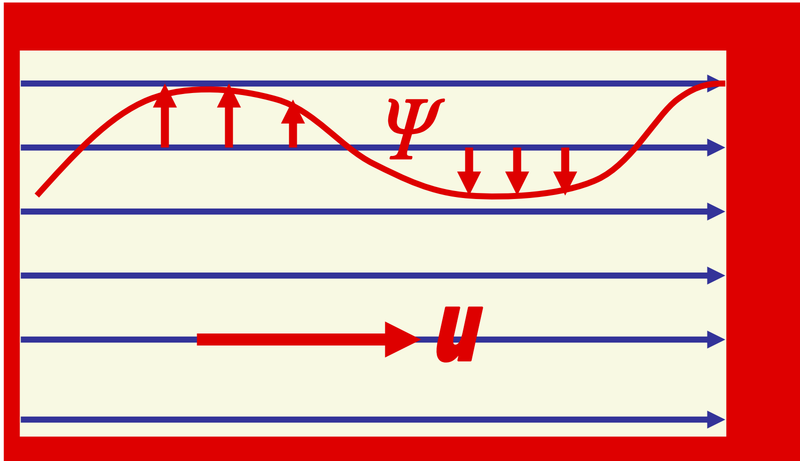
$$\frac{d\psi}{dt} = F(\psi) = \mathbf{L}\,\psi$$

$$\frac{d\psi}{dt} = \frac{\mathbf{i}}{t_0}\,\psi$$

$$\psi(\bullet) = \psi(x, t)$$

$$\psi(\times) = \psi(x - U\Delta t/2, t - \Delta t/2)$$

$$\psi(*) = \psi(x - U\Delta t, t - \Delta t)$$



Thus, in discrete presentation we will have

$$\frac{\psi(x,t)-\psi(x-U\Delta t,t-\Delta t)}{\Delta t}=$$

$$\frac{i}{2t_0}\begin{cases} 2\psi(x-U\Delta t/2,t-\Delta t/2), expl. \\ \psi(x,t)+\psi(x-U\Delta t,t-\Delta t), impl. \end{cases}$$

Looking for solution in the form of single harmonic wave

$$\psi(x,t)=\psi_0 \exp[\mathbf{i}(kx-\nu t)],$$

where

$$\psi_0 - amplitude, k - wavenumber, \nu - frequency$$

And using

$$\psi(x-U\Delta t, t-\Delta t)=\psi(x,t)e^{-\mathbf{i}\alpha},$$

$$\psi(x-U\Delta t/2, t-\Delta t/2)=\psi(x,t)e^{-\mathbf{i}\alpha/2},$$

$$\alpha=(kU-\nu)\Delta t$$

We get for frequency an equation

$$\frac{1-e^{-\mathbf{i}\alpha}}{\Delta t}=\frac{\mathbf{i}}{2t_0}\begin{cases}2e^{-\mathbf{i}\alpha/2}, & expl.\\ 1+e^{-\mathbf{i}\alpha}, & impl.\end{cases}$$

or

$$\frac{\Delta t}{2t_0}=\begin{cases}\sin(\alpha/2), & expl.\\ \tan(\alpha/2), & impl.\end{cases}$$

$$\alpha=(kU-\nu)\Delta t$$

$$\frac{\Delta t}{2t_0} = \begin{cases} \sin(\alpha/2), \ expl. \\ \tan(\alpha/2), \ impl. \end{cases}$$

$$\alpha = (kU - \nu)\Delta t$$

As |sin a| <= 1, |tan a| < ∞, we can conclude:

Explicit scheme is stable for

$$\Delta t \le 2t_0 \, ,$$

whereas the implicit scheme is unconditonally stable

Advantages of SISL

Large numerical stability, which results in

- **Increased time-step**
- **Increased modelling area of LAMs**
- **Enhanced resolution**

Coarse estimation: ***1 km of resolution -> 1 min in time step (experience with NH SISL HIRLAM):***

Δx=20-40 km:	***Δt=15 min;***
Δx=10 km:	***Δt=10 min;***
Δx= 3 km:	***Δt=3 min;***
Δx= 1 km:	***Δt=1 min;***

However, these estimates hold for predominantly horizontal (advective) dynamics.

Disadvantages of SISL
(if any)

- Multiple interpolations act as spectral smoothing:
  - Smoothing of fine details
  - Weak extra energy dissipation
  - No additional spectral filtration required
- Requires solution of an elliptic equation, which is a time-consuming procedure

  [All SI approaches have the same deficiency;

  The so-called split-explicit schemes are free of the shortcoming, yet possess another time-consuming mechanism do to the existence of the small ‘internal’ time step]

Unresolved problems

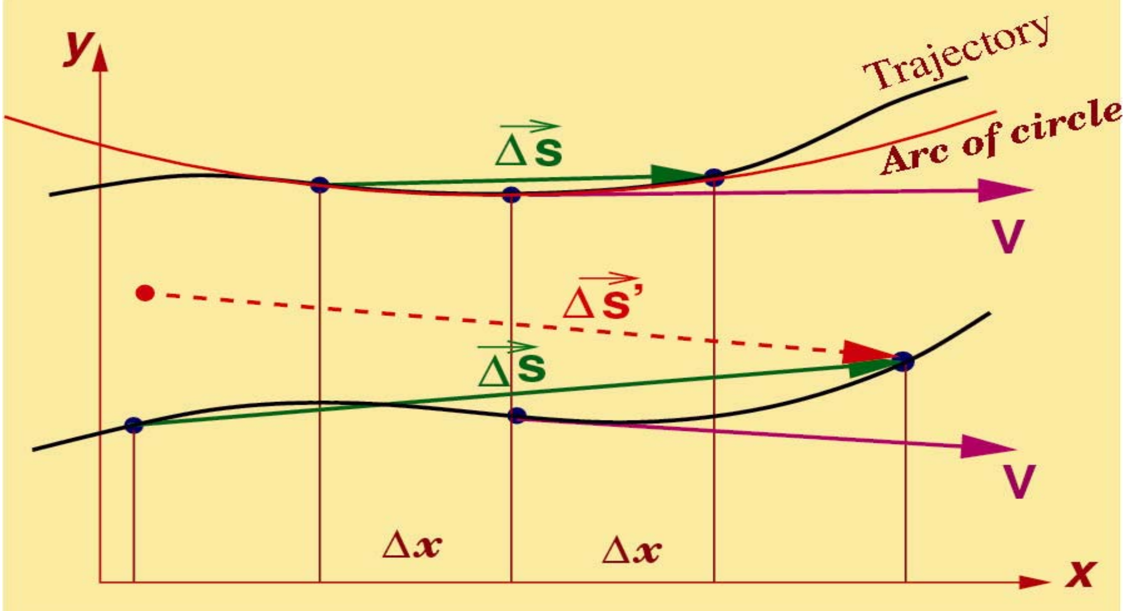
There is no SISL experience with (strong explicit) convection modelling:

We don't know, how useful is SISL at, say, 100 m horizontal and vertical resolutions, when vertical velocity becomes large (≥1 m/s)

It is expected, however, that SISL will operate 'normally', if condition

$$\Delta t \leq 2\, \Delta L/U,$$

where ΔL is interval between nodal points and U is 3D wind-speed, see the next slide.



Upper case:

$$\left|\Delta \mathbf{s}\right| = U\Delta t \le 2\Delta x,$$
$$\Delta \mathbf{s} \parallel \mathbf{V}$$

Trajectory calculation is correct

Lower case:

$$\left|\Delta \mathbf{s}\right| = U\Delta t \approx 4\Delta x(\ge 2\Delta x),\ \ \Delta \mathbf{s}\ not \parallel \mathbf{V},$$

Calculated trajectory **Δs'** is wrong!

Restriction to

the time step:

$$U\Delta t \le 2\Delta x$$

which gives **Δt = 1 min**. for **Δx= 1000 m, U = 30 m/s**

THE END