Non-hydrostatic Modelling
- Basic Equations -
- Simplifications -
- Vertical Coordinates -

Hans-Joachim Herzog

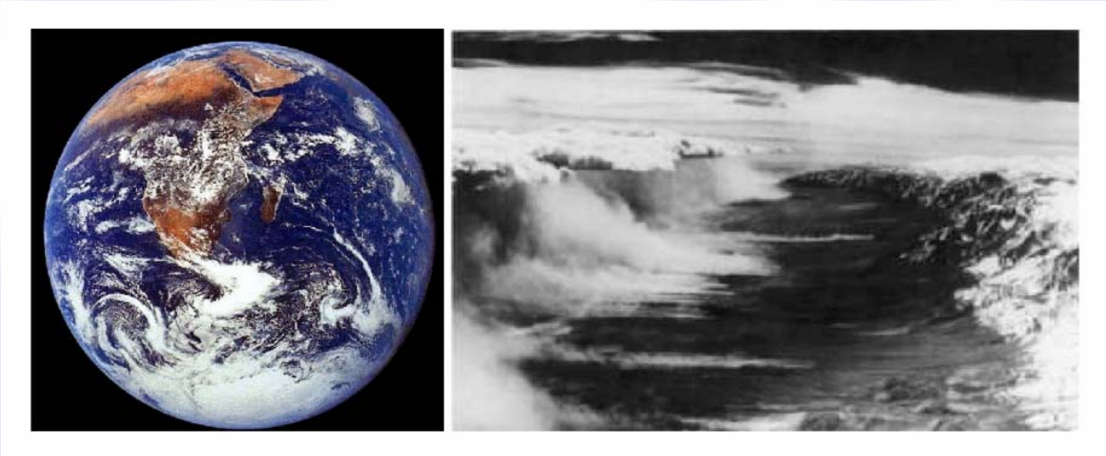
DWD - Potsdam , Germany

St.Petersburg Summer School
Sestroretsk , 11.-17.6.2006

1. BASIC EQUATIONS - physical basis

Budget Equations for Momentum , Mass , Heat , Water Components

constitute a model describing the impact of gravity and Earth rotation over an enormously wide spectral range of internal processes caused by heat, mass, momentum, radiation transfer and phase changes of water essentially determined by turbulence.



1.1 Coordinate-free basic equations

$$\rho \frac{d\vec{v}}{dt} = -\nabla p - \rho \nabla \phi - \rho\, 2\vec{\Omega} \times \vec{v} - \nabla \cdot \underline{\underline{T}}$$

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{v}$$

$$\rho \frac{de}{dt} = -p \nabla \cdot \vec{v} - \nabla \cdot \left(\vec{J}_e + \vec{R} \right) + \varepsilon \quad \Leftrightarrow \quad \rho \frac{dh}{dt} = \frac{dp}{dt} - \nabla \cdot \left(\vec{J}_e + \vec{R} \right) + \varepsilon$$

$$\rho \frac{dq^k}{dt} = -\nabla \cdot \vec{J}^k + I^k \qquad ; \quad \mathrm{k = v , l , f}$$

Doms, G. et al.,2002 : LM documentation Part I: Dynamics and Numerics (www.cosmo-model.org)

$\vec{v}$ - barycentric velocity

$\underline{\underline{T}}$ - molecular stress tensor

ε - molecular dissipation

p - pressure

ϕ - gravitational potential

$\rho = \sum_k \rho^k$ - total density (k = d , v , l , f)

$\vec{\Omega}$ - Earth rotation vector

$q^k = \frac{\rho^k}{\rho}$ - mass fraction (specific content) of constituent k

$\vec{R}$ - radiation flux vector

$e\,(h)$ - specific internal energy (enthalpy) with (h = e + p / ρ)

$\vec{J}_e \; \left(\vec{J}^k \right)$ - diffusion flux of heat (of q^k)

I^k - source / sink of q^k

1.2 Choice of basic equations dependent on a following numerical scheme

momentum equations

total mass equation (continuity equation)
enthalpy equation
water constituent equations

\+

equation of state

main-stream approach

elimination of density

Prognostic equation for pressure
instead of total density
continuity equation is hidden !

1.3 Common physical approximations

- original budget equations formulated for mean flow by Reynolds averaging approach (van Mieghem 1973)
- molecular fluxes, dissipation and almost all molecular diffusion fluxes are neglected compared to turbulent flux terms
- latent heat of vaporation and sublimation assumed constant
- specific heats of moist air are replaced by specific heat of dry air

turbulence averaging symbols dropped

1.3 Common physical approximations

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \rho \nabla \phi - \rho 2\vec{\Omega} \times \vec{v} - \nabla \cdot \overline{\left(\rho \vec{v}'' \vec{v}''\right)}$$

$$\rho c_{pd} \frac{dT}{dt} = \frac{dp}{dt} + \underline{Q_h}$$

$$\frac{dp}{dt} = -\frac{c_{pd}}{c_{vd}} p \nabla \cdot \vec{v} + \left(\frac{c_{pd}}{c_{vd}} - 1\right) \underline{Q_h} + \frac{c_{pd}}{c_{vd}} \underline{Q_m}$$

$$\rho \frac{dq^v}{dt} = -\nabla \cdot \vec{F}^v - \left(I^l + I^f\right)$$

$$\rho \frac{dq^{l,f}}{dt} = -\nabla \cdot \left(\vec{P}^{l,f} + \vec{F}^{l,f}\right) + I^{l,f}$$

$$\rho = p\left[R_d\left(1 + \left(\frac{R_v}{R_d} - 1\right) q^v - q^l - q^f\right) T\right]^{-1} = p\left(R_d T_v\right)^{-1}$$

<u>we follow this line</u>

(e.g. LM and MM5)

Doms et al. 2002

Dudhia 1993, MWR 121

$$T \rightarrow \theta = \frac{T}{\pi}$$

$$p \rightarrow \pi = \left(\frac{p}{p_{00}}\right)^{\frac{R_d}{c_{pd}}}$$

Bryan and Fritsch 2002, MWR 130

- overview article -

Definition of heat source term

$$Q_T := \frac{1}{\rho\, c_{pd}} Q_h$$

$$Q_T = -\frac{1}{\rho\, c_{pd}} \nabla \cdot \left(\vec{H} + \vec{R} \right) + \left(\frac{L_v}{c_{pd}} S^l + \frac{L_s}{c_{pd}} S^f \right)$$

Turbulent heat flux

Radiation flux

Diabatic heating due to cloud microphysical sources per unit mass

$$-\frac{1}{\rho\, c_{pd}} \nabla \cdot \vec{H} := -\frac{1}{\rho\, c_{pd}} \nabla \cdot c_{pd}\, \overline{\rho \vec{v}'' T}$$

$$\approx -\frac{1}{\rho\, c_{pd}} \nabla \cdot c_{pd}\, \overline{\pi}\, \overline{\rho}\, \overline{\vec{v}' \theta'} \approx \frac{1}{\rho} \frac{\partial}{\partial z} \left(\pi \rho K_h \frac{\partial \theta}{\partial z} \right)$$

$$S^l = I^l / \rho$$

$$S^f = I^f / \rho$$

Definition of moisture source term

$$Q_M := \frac{1}{\rho\, c_{pd}} Q_m$$

$$Q_M = \frac{R_d T}{\rho\, c_{pd}} \left(1 - \frac{R_v}{R_d} \right) \nabla \cdot \vec{F}^v - \frac{R_v T}{c_{pd}} \left(S^l + S^f \right) + \frac{R_d T}{\rho\, c_{pd}} \left[\nabla \cdot \left(\vec{F}^l + \vec{F}^f \right) + \nabla \cdot \left(\vec{P}^l + \vec{P}^f \right) \right]$$

Cloud heat sources

Turbulent flux for water constituents

$$\vec{F}^x = \overline{\rho \vec{v}'' q^x} \approx \overline{\rho\, w'' q^x}\, \vec{k} \approx -\rho K_h \frac{\partial q^x}{\partial z} \vec{k}$$

$$x = v, l, f$$

Precipitation (gravitational diffusion) fluxes

$$\vec{P}^{l,f} = -q^{l,f} \left| \vec{v}_T^{l,f} \right| \vec{k}$$

1.4 Mass-consistent formulation of T - and p – prognostic equations

$$\frac{dT}{dt} = \frac{1}{\rho c_{pd}} \frac{dp}{dt} + Q_T$$

$$\frac{dp}{dt} = -\frac{c_{pd}}{c_{vd}} p \nabla \cdot \vec{v} + \left(\frac{c_{pd}}{c_{vd}} - 1 \right) \rho c_{pd} Q_T + \frac{c_{pd}}{c_{vd}} \rho c_{pd} Q_M$$

source terms in prognostic p-equation often neglected

$$\frac{dT}{dt} = -\frac{p}{\rho c_{vd}} \nabla \cdot \vec{v} + Q_T$$

$$\frac{dp}{dt} = -\frac{c_{pd}}{c_{vd}} p \nabla \cdot \vec{v}$$

Dudhia 1993 , Doms et al. 2002

mass-consistent approach

$$\frac{dT}{dt} = -\frac{p}{\rho c_{vd}} \nabla \cdot \vec{v} + \frac{c_{pd}}{c_{vd}} \left(Q_T + Q_M \right)$$

$$\frac{dp}{dt} = -\frac{c_{pd}}{c_{vd}} p \nabla \cdot \vec{v} + \left(\frac{c_{pd}}{c_{vd}} - 1 \right) \rho c_{pd} Q_T + \frac{c_{pd}}{c_{vd}} \rho c_{pd} Q_M$$

up to here coordinate-free –
parameterisation problem dropped in the lecture

1.5 Spherical coordinates formulation of adiabatic dry model part

Equation system to start from:

$$\boxed{\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\nabla p - \nabla\phi - 2\vec{\Omega}\times\vec{v}}$$

$$\boxed{\frac{d\rho}{dt} = -\rho\nabla\cdot\vec{v}}$$

$$\boxed{\frac{d\theta}{dt} = 0}$$

$$\frac{dT}{dt} = -\frac{p}{c_v\rho}\nabla\cdot\vec{v}$$

$$\frac{dp}{dt} = -\frac{c_p}{c_v}p\nabla\cdot\vec{v}$$

$$\theta = T\left(\frac{p_{oo}}{p}\right)^{\frac{R}{c_p}}$$

$$p = R_d\,\rho T$$

introducing spherical coordinates

longitude	λ
latitude	φ
Radial distance from Earth centre	r

how to do that ?

Deep-Atmospheric Non-hydrostatic Equations for a Rotating Spherical Atmosphere

Lagrangian formalism from Theoretical Mechanics applied

Newton's second law of motion	$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = -\frac{1}{\rho}\frac{\partial p}{\partial q_k}$$
continuity equation	$$\frac{1}{\rho \mathbf{D}}\frac{d}{dt}(\rho \mathbf{D}) + \frac{\partial \dot{q}_k}{\partial q_k} = 0$$
first law of thermodynamics	$$\frac{\partial \theta}{\partial t} + \dot{q}_k \frac{\partial \theta}{\partial q_k} = 0$$

$$k = 1, 2, 3$$

$q_k \, , \dot{q}_k$ ***- generalised coordinates***

$L = T - \phi_N$ ***- Lagrangian function***

$T = \frac{1}{2}\vec{v}^2 = \frac{1}{2}\left(\frac{d\vec{r}}{dt}\right)^2$ ***- kinetic energy***

$\mathbf{D}^2 = \left|\frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k}\right|$ ***functional determinant squared***

$(d\vec{r})^2$ ***- metric form***

$\phi_N = \phi_N(q_k)$ ***- Newtonian gravitational potential***

Specifications

$$q_k = \lambda, \varphi, r \qquad ; \qquad \dot{q}_k = \dot{\lambda} + \Omega \, , \, \dot{\varphi} \, , \, \dot{r}$$

$$d\vec{r} = \left(r \cos\varphi \, d\lambda \, , \, r \, d\varphi \, , \, dr \right)$$

$$T = \frac{1}{2}\left(\frac{d\vec{r}}{dt}\right)^2 = \frac{1}{2}\left(r^2 \cos^2\varphi \left(\dot{\lambda} + \Omega\right)^2 + r^2 \dot{\varphi}^2 + \dot{r}^2 \right)$$

$$L = \frac{1}{2}\left(r^2 + \cos^2\varphi \left(\dot{\lambda}^2 + 2\Omega\,\dot{\lambda}\right) + r^2\dot{\varphi}^2 + \dot{r}^2 \right) - \left(\phi_N - \frac{1}{2} r^2 \cos^2\varphi \, \Omega^2 \right)$$

$$\boldsymbol{D}^2 = \begin{vmatrix} \frac{\partial^2 T}{\partial \dot{\lambda}^2} & 0 & 0 \\ 0 & \frac{\partial^2 T}{\partial \dot{\varphi}^2} & 0 \\ 0 & 0 & \frac{\partial^2 T}{\partial r^2} \end{vmatrix} = r^4 \cos^2\varphi$$

$$\phi = \phi_N - \frac{1}{2} r^2 \cos^2\varphi \, \Omega^2$$

geopotential = gravity potential = Newtonian potential + centrifugal potential

we define : $u := r\cos\varphi\,\dot{\lambda} \; , \; v := r\,\dot{\varphi} \; , \; w := \dot{r}$

deep equations

momentum equations

$$\frac{du}{dt} - 2\Omega \sin\varphi\, v + \underline{2\Omega \cos\varphi\, w} + \frac{u\,v}{r}\tan\varphi + \frac{u\,w}{\underline{r}} = -\frac{1}{\rho\, r \cos\varphi}\frac{\partial p}{\partial \lambda} - \frac{1}{r\cos\varphi}\frac{\partial \phi}{\partial \lambda}$$

$$\frac{dv}{dt} + 2\Omega \sin\varphi\, u + \frac{u^2}{r}\tan\varphi + \frac{v\,w}{\underline{r}} = -\frac{1}{\rho\, r}\frac{\partial p}{\partial \varphi} - \frac{1}{r}\frac{\partial \phi}{\partial \varphi}$$

$$\frac{dw}{dt} - \underline{2\Omega \cos\varphi\, u} \underline{- \frac{u^2 + v^2}{r}} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\partial \phi}{\partial r}$$

continuity equation

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{r\cos\varphi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \varphi}\left(v\cos\varphi\right)\right) + \frac{\partial w}{\partial r} + \underline{\frac{2}{r}w} = 0$$

first law of thermodynamics

$$\frac{d\theta}{dt} = \frac{\partial \theta}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial \theta}{\partial \lambda} + \frac{v}{r}\frac{\partial \theta}{\partial \varphi} + w\frac{\partial \theta}{\partial r} = 0 \cup \neq 0$$

relevant for 'Unified Model' of UK Met Office!

Literature:
Hinkelmann,K.H.: Primitive equations. WMO Training Seminar. Moscow 1965, pp.306-375.
White,A.A. et al.: Consistent approximate models of the global atmosphere... Q.J.R.Meteorol.Soc. 2005, 131, pp.2081-2107.

Consistent shallowness-approximation via the Lagrange route

spherical geometry : a - ***constant Earth radius***

$$\curvearrowright \quad r = a + z \qquad (\ z \ \text{- variable} \ )$$

$$\curvearrowright \quad \frac{\partial \phi}{\partial \lambda} = \frac{\partial \phi}{\partial \varphi} = 0 \ , \quad \frac{\partial \phi}{\partial r} \equiv \frac{\partial \phi}{\partial z} \Rightarrow g \approx const.$$

shallowness : $a >> z$

$$q_k = \lambda , \varphi , a + z \quad , \quad \dot{q}_k = \dot{\lambda} , \dot{\varphi} , \dot{z}$$

$$d\vec{r} = \left(a \cos\varphi \, d\lambda , a \, d\varphi , dz \right)$$

$$T = \frac{1}{2}\left(a^2 \cos^2\varphi \left(\dot{\lambda} + \Omega \right)^2 + a^2 \dot{\varphi}^2 + \dot{z}^2 \right) \quad , \quad \phi = \phi_N - \frac{1}{2} a^2 \cos^2\varphi \, \Omega^2$$

$$L = T - \phi \quad , \quad \mathbf{D} = a^2 \cos\varphi \quad , \quad u := a \cos\varphi \, \dot{\lambda} , v := a \dot{\varphi} , w := \dot{z}$$

shallow spherical equation

momentum equations

$$\frac{du}{dt} - \left(2\Omega + \frac{u}{a\cos\varphi}\right)\sin\varphi\, v = -\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial \lambda}$$

$$\frac{dv}{dt} + \left(2\Omega + \frac{u}{a\cos\varphi}\right)\cos\varphi\, u = -\frac{1}{\rho a}\frac{\partial p}{\partial \varphi}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

continuity equation

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{a\cos\varphi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial (v\cos\varphi)}{\partial \varphi}\right) + \frac{\partial w}{\partial z} = 0$$

first law of thermodynamics

$$\frac{d\theta}{dt} = \frac{\partial \theta}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial \theta}{\partial \lambda} + \frac{v}{a}\frac{\partial \theta}{\partial \varphi} + w\frac{\partial \theta}{\partial z} = 0 \cup \neq 0$$

Comparison with deep equations indicates that the <u>shallowness approximation</u> , consisting of

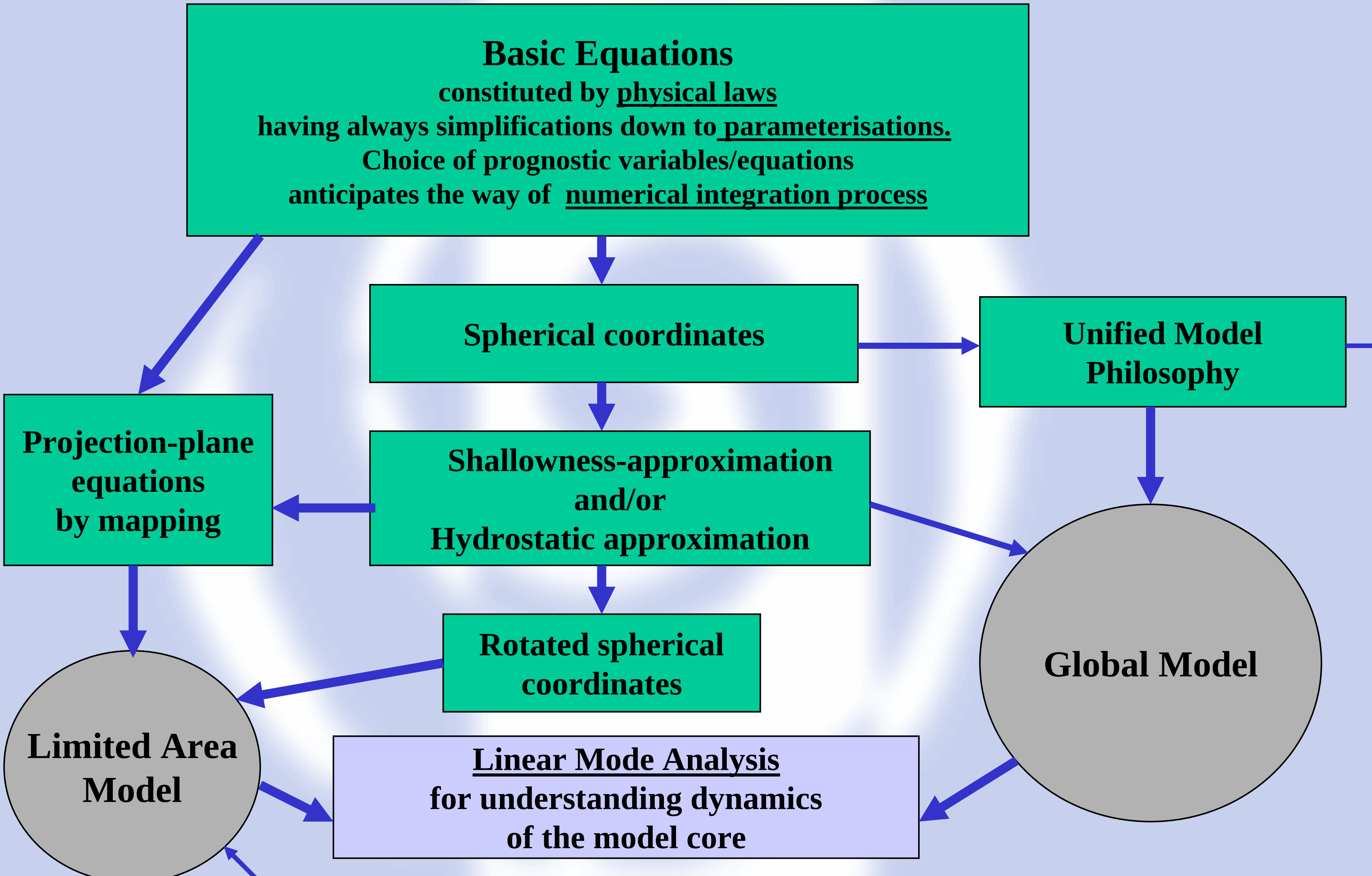
- **<u>'traditional approximation'</u> (neglecting the** $2\Omega\cos\varphi$ **-terms** $\longrightarrow$ **Eckart 1960)**

- **<u>'shallow - approximation'</u>** ($a >> z \, , \frac{\partial \phi}{\partial z} = const.$)

- **<u>'small-curvature approximation'</u> (neglecting** $\frac{u\,w}{r} , \frac{v\,w}{r} , \frac{u^2+v^2}{r}$ **in momentum equations, and** $\frac{2\,w}{r}$ **in the continuity equation) ,**

is achieved dynamically consistent as <u>one</u> package with the Lagrangian approach .

Some simplified overview



2. Linear Mode Analysis

Importance and meaning of a linear mode analysis is to recognise essential properties of the compressible non-hydrostatic model equations and understand simplifications due to filtering.
Why are the full equations important and interesting enough to become increasingly standard model equations?

The modes involved are

- **Rossby-, or advective mode**
- **Internal gravity (buoyancy)-inertial modes**
- **Acoustic modes**
- **Lamb mode**

Literature: **Miller, M. , 2002 : Atmospheric Waves.**
Meteorological Training Course Lecture Series, ECMWF.
Changnon, J. M. , P. R. Bannon , 2005 : Wave Response during Hydrostatic and Geostrophic Adjustment.
Part I : Transient Dynamics. *JAS, 62 , May 2005 , 1311-1329.*
Thuburn, J. , N. Wood, A. Staniforth , Normal modes of deep atmospheres.
2002 a : Spherical geometry. *Q.J.R.M.S. 128 , pp. 1771-1792.*
2002 b : f - F - plane geometry. Q.J.M.S. 128 , pp. 1793-1806.

2.1 Linear model framework

Linear Model framework → **Compressible , dry , inviscid , unforced , ideal gas , Cartesian coordinates , f - plane , invariant in y-direction , Linearisation on f - plane about constant basic current and height-variable basic state temperature with constant stability.**

Introduction of ‘ field variables ‘ in the sense of Eckart 1960
(cf. Gassmann and Herzog 2006)*

$$\left(u \;\; w \;\; T \right)^t := \sqrt{\rho_s \bar{\rho}} \left(u' \; w' \; c_p T' \right)^t \quad , \quad p := \sqrt{\frac{\rho_s}{\bar{\rho}}}\, p'$$

*** Gassmann and Herzog : A consistent Time-Split Numerical Scheme applied to the Nonhydrostatic Compressible Equations.**
- accepted to be published in: Monthly Wea.Rev. 2006

2.1 Linear model framework

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u-f\,v+\frac{\partial p}{\partial x}=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)v+f\,u=0$$

$$\underline{\mu}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)w+\left(\frac{\partial}{\partial z}+\Gamma\right)p-\frac{g}{c_p}\frac{\theta}{\overline{\theta}}=0$$

$$\frac{1}{c_s^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)p+\left(\frac{\partial}{\partial z}-\Gamma\right)w+\frac{\partial u}{\partial x}=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{g}{c_p}\frac{\theta}{\overline{\theta}}\right)+N^2 w=0$$

with the definitions

$\Gamma=-\frac{1}{2H}+\frac{g}{c_s^2}$ **- Eckart - coeffient**

$\underline{\mu}$ **- tracer**

$\frac{1}{H}=\frac{N^2}{g}+\frac{g}{c_s^2}$ **- reciprocal scale-height**

$N^2=\frac{g}{\overline{\theta}}\frac{d\overline{\theta}}{dz}$ **-Brunt-Väisälä frequency squared**

General case :

nonhydrostatic $\rightarrow \mu = 1$

compressible $\rightarrow c_s^2 < \infty$

Identical structural wave equation for both p and w

$$\boxed{\left(\tilde{D}^2 + f^2\right)\left(\frac{\partial^2}{\partial z^2} - \Gamma^2\right)\begin{pmatrix} p \\ w \end{pmatrix} + \left(\underline{\mu}\,\tilde{D}^2 + N^2\right)\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c_s^2}\left(\tilde{D}^2 + f^2\right)\right]\begin{pmatrix} p \\ w \end{pmatrix} = 0}$$

with the individual tendency operator $\tilde{D} \equiv \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)$

variable separation ansatz:

$$w \sim w_n(x,t)\,\psi_n^{(w)}(z) \quad , \quad p \sim p_n(x,t)\,\psi_n^{(p)}(z)$$

$$\frac{1}{h_n^2}\left(\tilde{D}^2 + f^2\right)\begin{pmatrix} w_n \\ p_n \end{pmatrix} + \left(\underline{\mu}\tilde{D}^2 + N^2\right)\left[\frac{1}{c_s^2}\left(\tilde{D}^2 + f^2\right) - \frac{\partial^2}{\partial x^2}\right]\begin{pmatrix} w_n \\ p_n \end{pmatrix} = 0$$

horizontal, time-dependent structural equation

$$\frac{d^2\psi_n^{(p,w)}}{d z^2} + \nu_n^2\,\psi_n^{(p,w)} = 0$$

vertical structural equation

2.2 Vertical structural solution

with vertical boundary condition : $w=0$ **for** $z=0 \cap z=z_T<\infty$

$$\boxed{\begin{aligned} &\frac{d^2\psi_n^{(p)}}{dz^2}+\nu_n^2\,\psi_n^{(p)}=0 \\ &BC: \frac{d\psi_n^{(p)}}{dz}+\Gamma\,\psi_n^{(p)}=0 \end{aligned}} \qquad \boxed{\begin{aligned} &\frac{d^2\psi_n^{(w)}}{dz^2}+\nu_n^2\,\psi_n^{(w)}=0 \\ &BC: \quad \psi_n^{(w)}=0 \end{aligned}}$$

$$\nu_n^2=\frac{n^2\pi^2}{z_T^2}>0$$ - ***vertical wavenumber squared***

$$\boxed{\psi_n^{(p)}(z)=\left(\frac{n\pi}{z_T}\right)\cos\left(\frac{n\pi}{z_T}z\right)-\Gamma\sin\left(\frac{n\pi}{z_T}z\right)} \qquad \boxed{\psi_n^{(w)}(z)=-\Gamma\sin\left(\frac{n\pi}{z_T}z\right)}$$

$$h_n^{-2}=\nu_n^2+\Gamma^2$$ - ***separation constant in horizontal structural equation***

$n=1,2,3,\dots$

$$\boxed{\psi_0^{(p)}=\exp(-\Gamma z)} \qquad \boxed{\psi_0^{(w)}=0}$$

$n=0$

$$\nu_0^2=-\Gamma^2 \qquad h_n^{-2}=0$$

2.3 Horizontal structural equation

wave solution

$$w_n(x,t) \sim \sin(kx - \omega_n t) \qquad p_n(x,t) \sim \cos(kx - \omega_n t)$$

frequency equation

$$\frac{\left[N^2 - \mu(kU - \omega_n)^2\right]\left[k^2 + \left(f^2 - (kU - \omega_n)^2\right)/c_s^2\right]}{\left[(kU - \omega_n)^2 - f^2\right]} = \begin{cases} \dfrac{1}{h_n^2} \\ \nu_n^2 + \Gamma^2 \\ \nu_n^2 + \dfrac{1}{4H^2} - \dfrac{N^2}{c_s^2} \end{cases}$$

Gossard and Hook , 1975 : Waves in the Atmosphere. p. 112, eq. (23.7)

All possible wave frequencies of the given linear model are contained in the frequency equation above. They are going to be discussed in the following.

Nonhydrostatic - compressible/elastic

$$\mu = 1 \quad c_s^2 < \infty$$

$$\underline{\mu}\frac{\sigma_n^4}{c_s^2} - \sigma_n^2\left(\underline{\mu} k^2 + \left(\nu_n^2 + \frac{1}{4H^2}\right) + \underline{\mu}\frac{f^2}{c_s^2}\right) + k^2 N^2 + \left(\nu_n^2 + \frac{1}{4H^2}\right) f^2 = 0$$

$$\underline{\mu} = 1$$

$$\sigma_n^2 := \left(kU - \omega_n\right)^2$$ **- intrinsic frequency**

$$\left(\sigma_n^2\right)_a \approx c_s^2\left(k^2 + \nu_n^2 + \frac{1}{4H^2}\right) + f^2 \approx c_s^2\left(k^2 + \nu_n^2 + \frac{1}{4H^2}\right)$$

pairs of internal acoustic mode frequencies

$$\left(\sigma_n^2\right)_g \approx \frac{k^2 N^2 + \left(\nu_n^2 + \frac{1}{4H^2}\right) f^2}{\left(k^2 + \nu_n^2 + \frac{1}{4H^2}\right) + \frac{f^2}{c_s^2}}$$

pairs of internal gravity mode frequencies

$$\boxed{\begin{array}{l}\mu = 1 \\ c_s^2 \to \infty\end{array}}$$

Nonhydrostatic - incompressible/anelastic

$\mu = 1 \quad c_s^2 \to \infty$

$$\mu \frac{\sigma_n^4}{c_s^2} - \sigma_n^2 \left(\mu k^2 + \left(\nu_n^2 + \frac{1}{4H^2} \right) + \mu \frac{f^2}{c_s^2} \right) + k^2 N^2 + \left(\nu_n^2 + \frac{1}{4H^2} \right) f^2 = 0$$

$c_s^2 \to \infty \qquad \mu = 1$

$$\left(\sigma_n^2\right)_g \approx \frac{k^2 N^2 + \left(\nu_n^2 + \frac{1}{4H^2} \right) f^2}{k^2 + \nu_n^2 + \frac{1}{4H^2}}$$

internal gravity modes remain untouched

acoustic mode filtered out due to $c_s^2 \to \infty$

$\mu = 0$

$c_s^2 < \infty$

Hydrostatic approximation - compressible/elastic

$$\mu = 0 \quad c_s^2 < \infty$$

$$\mu \frac{\sigma_n^4}{c_s^2} - \sigma_n^2 \left(\mu k^2 + \left(\nu_n^2 + \frac{1}{4H^2} \right) + \mu \frac{f^2}{c_s^2} \right) + k^2 N^2 + \left(\nu_n^2 + \frac{1}{4H^2} \right) f^2 = 0$$

$$\mu = 0 \qquad c_s^2 < \infty$$

$$\left(\sigma_n^2 \right)_g \approx \frac{k^2 N^2}{\nu_n^2 + \frac{1}{4H^2}} + f^2$$

internal gravity modes with limited validity

$$k^2 << \nu_n^2 + \frac{1}{4H^2}$$

$$L_x^2 >> L_z^2$$

- *acoustic mode filtered out due to hydrostatic approximation*
- *hydrostatic filtering, however, is insufficient to represent gravity waves with high-resolving models*
- *Motivation for nonhydrostatic modelling!*

2.4 Analytical solution

Nonstationary solutions for linear cases considered above

$$u(x,z,t) \sim \frac{k(kU-\omega_n)}{f^2-(kU-\omega_n)^2}\cos(kx-\omega_n t)\underline{\psi_n^{(p)}(z)}$$

$$v(x,z,t) \sim -\frac{k\,f}{f^2-(kU-\omega_n)^2}\sin(kx-\omega_n t)\underline{\psi_n^{(p)}(z)}$$

$$p(x,z,t) \sim \cos(kx-\omega_n t)\underline{\psi_n^{(p)}(z)}$$

$$\frac{g}{c_p}\frac{\theta(x,z,t)}{\overline{\theta}(z)} \sim \frac{N^2}{kU-\omega_n}\cos(kx-\omega_n t)\underline{\psi_n^{(w)}(z)}$$

$$w(x,z,t) \sim \sin(kx-\omega_n t)\underline{\psi_n^{(w)}(z)}$$

$$\underline{(kU-\omega_n)\neq 0}$$

The vertical structural function $\underline{\psi_n^{(p)}(z)}\left(\underline{\psi_n^{(w)}(z)}\right)$ *is sufficient to represent the vertical structure of the variables* $\underline{u,v,p}\ \left(\underline{w,\theta}\right)$ *.*

This finding suggests for a vertical difference approximation the application of a Charney-Phillips (CP-) grid instead of a Lorenz (L-) grid!

The case

$$n = 0 \longrightarrow \psi_0^{(p)} = \exp\left(-\Gamma z\right) \qquad \psi_0^{(w)} = 0$$

with $$\nu_0^2 = -\Gamma^2 \qquad h_n^{-2} = 0$$

defines the Lamb mode which is a particular solution of the vertical structural equations with boundary conditions included. Its frequency is

$$\sigma_o^2 = \left(kU - \omega_0\right)^2 \approx k^2\, c_s^2$$

indicating a horizontally propagating acoustic wave evanescent in vertical direction. For a pure Lamb wave

$w = 0$ ***and*** $\theta = 0$ ***is valid.***

2.5 Simplified scheme towards filtered equations

hydrostatic equations

$$\tilde{D}u - f\,v + \frac{\partial p}{\partial x} = 0$$

$$\tilde{D}v + f\,u = 0$$

$$\left(\frac{\partial}{\partial z} + \Gamma\right)p - \frac{g}{c_p}\frac{\theta}{\overline{\overline{\theta}}} = 0$$

$$\frac{1}{c_s^2}\tilde{D}\,p + \left(\frac{\partial}{\partial z} - \Gamma\right)w + \frac{\partial u}{\partial x} = 0$$

$$\tilde{D}\left(\frac{g}{c_p}\frac{\theta}{\overline{\overline{\theta}}}\right) + N^2 w = 0$$

full equations

$$\tilde{D}u - f\,v + \frac{\partial p}{\partial x} = 0$$

$$\tilde{D}v + f\,u = 0$$

$$\underline{\mu}\,\tilde{D}w + \left(\frac{\partial}{\partial z} + \Gamma\right)p - \frac{g}{c_p}\frac{\theta}{\overline{\overline{\theta}}} = 0$$

$$\frac{1}{\underline{c_s^2}}\tilde{D}\,p + \left(\frac{\partial}{\partial z} - \Gamma\right)w + \frac{\partial u}{\partial x} = 0$$

$$\tilde{D}\left(\frac{g}{c_p}\frac{\theta}{\overline{\overline{\theta}}}\right) + N^2 w = 0$$

anelastic equations

$$\tilde{D}u - f\,v + \frac{\partial p}{\partial x} = 0$$

$$\tilde{D}v + f\,u = 0$$

$$\tilde{D}w + \left(\frac{\partial}{\partial z} + \Gamma\right)p - \frac{g}{c_p}\frac{\theta}{\overline{\overline{\theta}}} = 0$$

$$\left(\frac{\partial}{\partial z} - \Gamma\right)w + \frac{\partial u}{\partial x} = 0$$

$$\tilde{D}\left(\frac{g}{c_p}\frac{\theta}{\overline{\overline{\theta}}}\right) + N^2 w = 0$$

$$\underline{\tilde{D}w \Rightarrow 0} \qquad \underline{\tilde{D}p \Rightarrow 0}$$

$$\left.\begin{aligned}
&\tilde{D}\left(\frac{\partial u}{\partial x}\right) - f\frac{\partial v}{\partial x} + \frac{\partial^2 p}{\partial x^2} = 0\\
&\tilde{D}\left(\frac{\partial v}{\partial x}\right) + f\frac{\partial u}{\partial x} = 0\\
&\tilde{D}\left(\frac{\partial^2}{\partial z^2} - \frac{1}{4H^2}\right)p - N^2\frac{\partial u}{\partial x} = 0
\end{aligned}\right\} \quad \underline{\tilde{D}\left(\frac{\partial u}{\partial x}\right) \Rightarrow 0}$$

$$\boxed{\tilde{D}\left\{\frac{\partial^2 p}{\partial x^2} + \frac{f^2}{N^2}\left(\frac{\partial^2}{\partial z^2} - \frac{1}{4H^2}\right)p\right\} = 0}$$

geostrophic potential vorticity equation
- advective mode only

with $$\frac{N^2}{f^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} - \frac{1}{4H^2}w = 0$$

Nonhydrostatic model equations

Compressible/elastic	Incompressible/anelastic *	Hydrostatic (primitive) equations
equations with full physical structure	reduction of equation set due to $\frac{dp}{dt}=0 \rightarrow \left(\frac{\partial}{\partial z}-\Gamma\right)w+\frac{\partial u}{\partial x}=0$	reduction to hydrostatic equation $\frac{dw}{dt}=0 \rightarrow \left(\frac{\partial}{\partial z}+\Gamma\right)p-\frac{g}{c_p}\frac{\theta}{\overline{\theta}}=0$
full set of prognostic variables	filter condition: $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p-\Gamma^2 p=F(x,z);\ \forall t$ boundary value problem constitutes a diagnostic relation for p	filter condition: $\frac{\partial^2 w}{\partial z^2}-\frac{1}{4H^2}w=G(x,z);\ \forall t$ boundary value problem constitutes a diagnostic relation for w
Internal acoustic and gravity waves are completely contained. Hydrostatic and geostrophic adaptation	Internal gravity waves are completely contained, acoustic waves are filtered out. Hydrostatic and geostrophic adaptation	Internal gravity waves are contained, but insufficiently presented $\left(L_x^2 >> L_z^2\right)$ acoustic waves excluded. Geostrophic adaptation
Damping/filtering of acoustic waves due to an appropriate numerical scheme (split-explicit , semi-implicit-semi-Lagrange) Models: MM5, LM, WRF-NCAR, UK-Unified Model, etc. ... WK78 , Cullen, Gassmann	Numerical treatment of elliptic pressure equation is difficult and needs to be done with care (terrain-following coordinate). An excellent research model is EULAG (Grabowski, W.W.,P.K.Smolarkiewicz, 2002,MWR 130, 939-952.)	Most operational global and limited area models are hydrostatic.

* refinements of anelastic approximation: P.B.Bannon,1996, J.Atm.Sc. 53, No.23, 3618-3628.

3. Vertical coordinates

The choice of an appropriate vertical coordinate is always to aim at improving simulations over mountainous terrain

- *Introduction of terrain-following vertical coordinate in meteorological modelling by Phillips 1957 , Gal-Chen and Somerville 1975 , applying a terrain-following normalisation with surface pressure (time-dependent coordinate -> deformable) or surface-height (time-independent coordinate -> nondeformable)*
  - *Phillips' sigma-coordinate -> larger-scale hydrostatic modelling*
  - *Gal-Chen's coordinate -> small-scale nonhydrostatic modelling*
- *There are problems in computing the pressure gradient term with pressure coordinate in hydrostatic models in case of steeper mountains which has led to the introduction of step-terrain orography (Mesinger 1984, Mesinger et al. 1985, 1988 -> NCEP Regional Eta Model), which seems however not appropriate in nonhydrostatic modelling (Gallus and Klemp 2000).*
- *Revival of Z-coordinate as shaved-cell approach in combination with finite-volume method – a possible breakthrough ?*

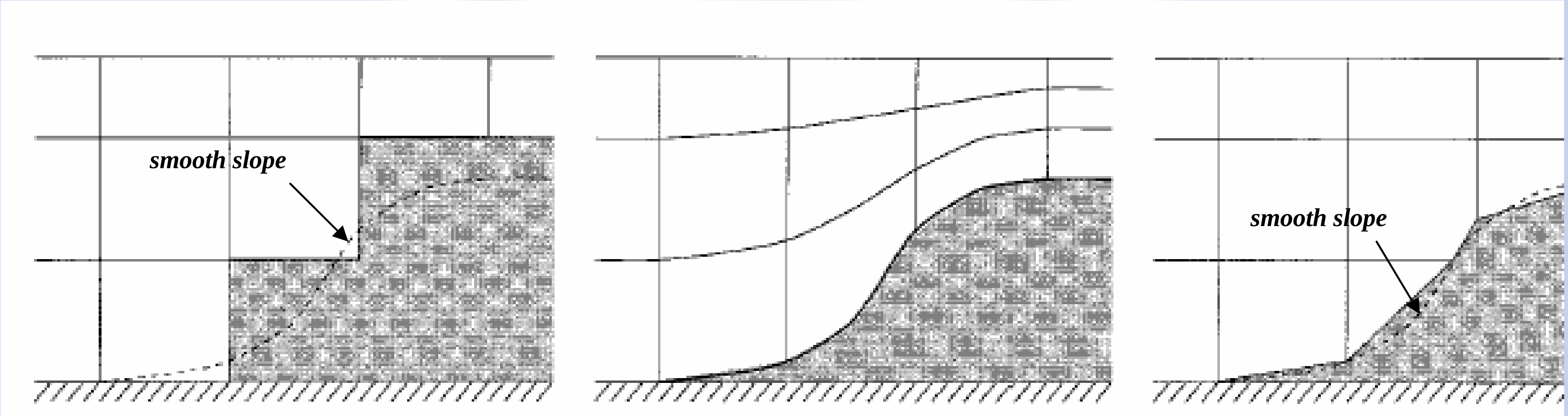
3.1 Representation of mountains in nonhydrostatic models

Revival of Z-coordinate in nonhydrostatic modelling !

step topography
in Z- coordinate model

terrain-following
coordinate model

Z-coordinate model
with piecewise constant slopes
(shaved-cell approach)



Bonaventura, L. , 2000:
J. of Comput. Phys. 158, 186-213.

Gallus, A. W., J. B. Klemp, 2000:
MWR 128 , 1153-1164.

research state !

Applied in several non-hydrostatic models even in operational mode
(LM , MM5 , UK , etc.)

Adcroft, A. et al. , 1997:
MWR 125 , 2293-2315.
(for Ocean Modelling)
Steppeler, J. et al. , 2002:
MWR 130 , 2143-2149.
(application in LM)

research state !

3.2 Introduction of a time-independent terrain-following coordinate

- *Vertical coordinate ζ may be any monotonic function of geometrical height.*
- *This ζ -system is fixed in physical space, and is non-orthogonal.*
- *It is of Gal-Chen type.*
- *The lowest coordinate surface of constant ζ coincides with the smooth model orography.*
- *The lower boundary condition seems easy but should be formulated carefully and consistent which is actually not simple to be done.*

How to transform the basic equations into such a non-orthogonal ζ - system?

Method :
Start from the basic equations in covariant vector form

3.3 Transformation method

Brief survey of the method

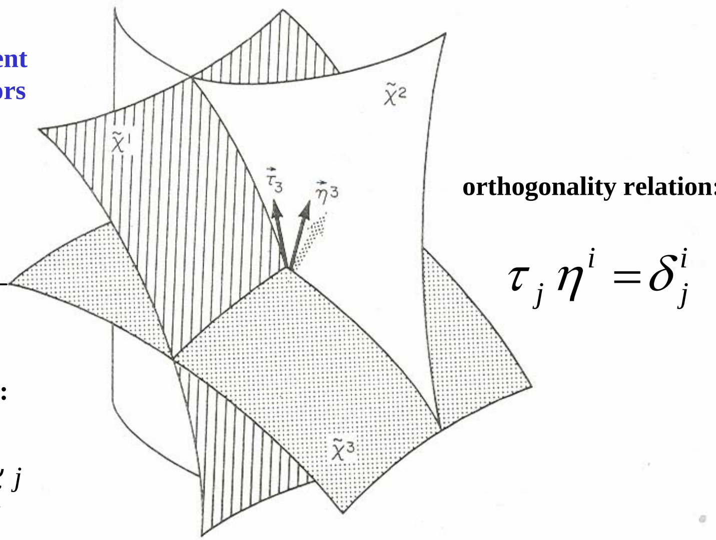
two different basis vectors

covariant:

$$\tau_j = \frac{\partial \vec{r}}{\partial \tilde{x}^j}$$

contravariant:

$$\eta^j = \nabla \tilde{x}^j$$



orthogonality relation:

$$\tau_j \eta^i = \delta^i_j$$

Fig. 6-1. Illustration of the two types of basis vectors in a nonorthogonal coordinate representation. The vector η^3 is perpendicular to the plane $\tilde{x}^3$ = constant, whereas τ_3 is tangent to the curve along which each coordinate except $\tilde{x}^3$ is a constant (from Dutton, 1976).

$$\vec{v} = \tilde{u}^j \tau_j = \tilde{u}_j \eta^j$$

Momentum equations in contravariant form:

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^j \left(\frac{\partial \tilde{u}^i}{\partial \tilde{x}^j} + \tilde{\Gamma}^i_{jl} \tilde{u}^l \right) = -\tilde{G}^{ij} \frac{1}{\rho} \frac{\partial p}{\partial \tilde{x}^j} - \frac{\partial \tilde{x}^i}{\partial \tilde{x}^3} g - 2\tilde{\varepsilon}^{ijl} \tilde{\Omega}_j \tilde{u}_l$$

Divergence of wind vector:

$$\nabla \cdot \vec{v} = \frac{\partial \tilde{u}^i}{\partial \tilde{x}^i} + \tilde{\Gamma}^i_{i,s} \tilde{u}^s = \frac{1}{\sqrt{\tilde{G}}} \frac{\partial}{\partial \tilde{x}^i} \left(\sqrt{\tilde{G}}\, \tilde{u}^i \right)$$

Literature:

Dutton, J.A. , *1986 : The ceaseless wind. pp. 129-144 , 248-251*

Pielke, R.A. , *1984 : Mesoscale Meteorological Modeling. pp.102-127*

Gal-Chen, T. , R.C.J. Somerville , *1975 : On the use of a coordinate transformation for the solution of the Navier-Stokes equations. J.Comput.Phys. 17, 209-229*

Zdunkowski, W., A.Bott : *Dynamics of the Atmosphere.* *A course in Theoretical Meteorology.* **Cambridge Univ.Press 2003.**

3.3 Transformation method

further specifications

$$\tilde{G}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^l}\frac{\partial \tilde{x}^j}{\partial x^l} \qquad \tilde{G}_{ij} = \frac{\partial x^l}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j} \qquad \sqrt{\tilde{G}} = \left|\frac{\partial x^i}{\partial \tilde{x}^j}\right| \qquad \tilde{\Gamma}^i_{jl} = \tilde{\Gamma}^3_{jl} = \frac{\partial^2 z}{\partial \tilde{x}^j \partial \tilde{x}^l}\frac{\partial \zeta}{\partial z}$$

transformation

$$(x, y, z) \Longleftrightarrow (x, y, \zeta)$$

old set $$x^1,\ x^2,\ x^3 = x,\ y,\ z(x, y, \zeta)$$

new set $$\tilde{x}^1,\ \tilde{x}^2,\ \tilde{x}^3 = x,\ y,\ \underline{\zeta(x, y, z)}$$

monotonic in z !

$$\tilde{u}^1,\ \tilde{u}^2,\ \tilde{u}^3 = u,\ v,\ \dot{\zeta}$$

and

$$w = \tilde{u}^i \frac{\partial x^3}{\partial \tilde{x}^i}$$

with $$p(x,y,z,t) = \underline{p_0(z)} + p'(x,y,z,t) \quad , \quad z = z(x,y,\zeta)$$

hydrostatic vertical distribution of basic state pressure $p_0(z)$!

and the definition $$\underline{\sqrt{G}} := -\sqrt{\tilde{G}} = -\left|\frac{\partial x^i}{\partial \tilde{x}^j}\right| = \underline{-\frac{\partial z}{\partial \zeta} > 0}$$ *(left-handed system!)*

We arrive finally at the nonhydrostatic equations in the ζ *- system assuming a dry-adiabatic, unforced, inviscid model atmosphere, here for the sake of simplicity, in a* x, y, ζ *- system .*

A peculiarity of most models is to use a prognostic variable in the vertical momentum equation which is not the contravariant vertical wind component $\dot{\zeta}$ *but the common covariant variable* $w = \dot{z}$ *instead .*

Having

$$\underline{\frac{d}{dt}} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \underline{\frac{\partial}{\partial t} + u\left.\frac{\partial}{\partial x}\right|_\zeta + v\left.\frac{\partial}{\partial y}\right|_\zeta + \dot{\zeta}\frac{\partial}{\partial \zeta}}$$

$$\underline{\nabla \cdot \vec{v}} = \frac{1}{\sqrt{G}}\left(\frac{\partial \sqrt{G}u}{\partial x} + \frac{\partial \sqrt{G}v}{\partial y} + \frac{\partial \sqrt{G}\dot{\zeta}}{\partial \zeta}\right)$$

[*flux-form very important for model implementation!*]

3.4 Transformed equations

$$\frac{du}{dt} = -\frac{1}{\rho}\left(\left.\frac{\partial p'}{\partial x}\right|_{\zeta} + \frac{1}{\sqrt{G}} \left.\frac{\partial z}{\partial x}\right|_{\zeta} \frac{\partial p'}{\partial \zeta} \right) + f\,v$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\left(\left.\frac{\partial p'}{\partial y}\right|_{\zeta} + \frac{1}{\sqrt{G}} \left.\frac{\partial z}{\partial y}\right|_{\zeta} \frac{\partial p'}{\partial \zeta} \right) - f\,u$$

$$\frac{dw}{dt} = \frac{1}{\rho}\frac{1}{\sqrt{G}}\frac{\partial p'}{\partial \zeta} + \frac{g\,\rho_0}{\rho}\left(\frac{T'}{T} - \frac{T_0}{T}\frac{p'}{p_0} \right)$$

$$\frac{d\,p'}{dt} - g\,\rho_0\,w = -\frac{c_{pd}}{c_{vd}}\,p\,\nabla\cdot\vec{v}$$

$$\frac{dT}{dt} = -\frac{p}{c_{vd}\,\rho}\nabla\cdot\vec{v} \quad , \qquad p = R_d\,\rho\,T$$

Mind the pressure gradient terms !

Nonhydrostatic equations written with generalised terrain-following ζ - coordinate

Further introduction of spherical coordinates - with shallowness-approximation - leads to the dynamical core equations of the Lokal-Modell (LM)

relation between contravariant and physical vertical motion →

$$\dot{\zeta} = \frac{1}{\sqrt{G}}\left(u\left.\frac{\partial z}{\partial x}\right|_{\zeta} + v\left.\frac{\partial z}{\partial y}\right|_{\zeta} - w \right)$$

3.5 Lower boundary condition

… has an important practical implication. It should be formulated as consistent as possible within a given numerical scheme, where the scheme is also necessary to be adapted to this problem. A successful approach has been developed by Almut Gassmann.*

At terrain-following lower boundary (LB) → free-slip condition :

$$\boxed{\dot{\zeta}_{LB} = 0 \ , \ \forall t}$$

$$\dot{\zeta} = \frac{1}{\sqrt{G}} \left(u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial x} - w \right) \longrightarrow w_{LB} = \frac{1}{\sqrt{G}} \left(u \frac{\partial z}{\partial x} + v \frac{\partial z}{\partial x} \right)_{LB}$$

---

*Gassmann, A. , 2004 : Formulation of the LM's Dynamical Lower Boundary Condition. COSMO-Newsletter No.4, Febr. 2004, 155-158. (www.cosmo-model.org)

Gassmann, A. , H.-J. Herzog , 2006 : A consistent time-split numerical scheme applied to the nonhydrostatic compressible equations. MWR, accepted to be published.

3.5 Lower boundary condition

$$\frac{\partial \dot{\zeta}_{LB}}{\partial t} = 0 \longrightarrow \left(\frac{\partial w}{\partial t}\right)_{LB} = \frac{1}{\sqrt{G}}\left(\frac{\partial u}{\partial t}\frac{\partial z}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial z}{\partial x}\right)_{LB}$$

elimination of $\left(\frac{\partial w}{\partial t}\right)_{LB}$ ***by use of*** $\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{1}{\sqrt{G}}\frac{\partial p'}{\partial \zeta} + \frac{g\,\rho_0}{\rho}\left(\frac{T'}{T} - \frac{T_0}{T}\frac{p'}{p_0}\right) + f_w$

$$\frac{\partial p'}{\partial \zeta} = \rho\sqrt{G}\left[\frac{\partial z}{\partial x}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial v}{\partial t} + \frac{g\rho_0 T_0}{\rho T}\frac{p'}{p_0} - \frac{g\rho_0}{\rho}\frac{T'}{T} - f_w\right]$$

elimination of $\frac{\partial p'}{\partial \zeta}$

in u- and v-momentum equation

elimination of $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$

by use of u- and v-momentum equation

3.6 Upper boundary condition

- *Non-penetrative boundary condition at* $\zeta = 0$

  - *rigid lid* → $\boxed{\dot{\zeta} = w = 0}$ *(flat)*

  - *no mass flux across the boundary* → $\frac{\partial}{\partial \zeta}(u, v, T, \ldots) = 0$

Danger of wave reflection without additional absorbing remedies !
→Sponge technique (philosophy of Davies + Kallberg 1976, 1983)

- *Radiative upper boundary condition* $\boxed{\hat{p}' = \frac{\overline{\rho}\, N}{k}\, \hat{w}}$

It is possible to be implemented in a nonlinear nonhydrostatic model (LM) for real-data integrations, although resting on limited assumptions (linear, hydrostatic, incompressible →
Klemp, Durran 1983, and also Bougeault 1983)

demonstrated in Almut Gassmann's lecture

3.5 Lower boundary condition

Neumann boundary condition at ζ_{LB} →

$$\frac{\partial p'}{\partial \zeta} = \rho \sqrt{G} \left(\frac{\frac{\partial z}{\partial x} F_u + \frac{\partial z}{\partial y} F_v + B}{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \right)$$

$$\frac{\partial u}{\partial t} = \frac{\left(1 + \left(\frac{\partial z}{\partial y}\right)^2\right) F_u - \left(\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right) F_v - \frac{\partial z}{\partial x} B}{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \qquad \frac{\partial v}{\partial t} = \frac{\left(1 + \left(\frac{\partial z}{\partial x}\right)^2\right) F_v - \left(\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right) F_u - \frac{\partial z}{\partial y} B}{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

↑ ***relevant for implementation***

$$F_u = f_u + f\,v - \frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$F_v = f_v - f\,v - \frac{1}{\rho}\frac{\partial p'}{\partial y}$$

$$B = \frac{g\rho_0 T_0}{\rho T}\frac{p'}{p_0} - \frac{g\rho_0}{\rho}\frac{T'}{T} - f_w$$

V.BJERKNES , who was the first advocat of NWP, wrote 1904 :
' If it is true, as every scientist believes, that subsequent atmospheric states develop from the preceding ones to physical law, then it is apparent that the sufficient and necessary conditions for rational solution of forecasting problems are the following:
. . . A sufficiently accurate knowledge of the laws according to which one state of the atmosphere develops from another . . . The problem is of huge dimensions. Its solution can only be the result of a long development . . .'

1998, when A. ARAKAWA retired , he has reflected Bjerknes' famous note such : ' I will not be able to see the completion of the 'great challenge', but I am happy to see at least its beginning .'

Where we are now ?

5. Outlook

Outline of main points :

- *Increasing tendency towards unified global non-hydrostatic models*
  Tendency towards global gridpoint models
  Introduction of quasi-homogeneous, quasi-isotropic grids
  (geodesic grid : icosahedron → polyhedron)

- *New formulation principles in view of a Vortex-Energy Theory (Nambu-bracket theory) from P. Nevir (1993, 1998) → opens the way for the construction of new spatial difference schemes generalising the classic ideas from Arakawa for the Jacobian operator up to the general adiabatic nonhydrostatic compressible equations connecting both global and local accuracy . Recent suggestions for such new difference constructions have been made by R.Salmon (2005).*

Literature:

Heikes,R., D.A.Randall,1995: MWR 123,1881-1887.
Majewski,D. et al.,2002: MWR 130,319-338.
Salmon, R.,2005: Nonlinearity 18,R1-R16.
Nevir,P.,R.Blender,1993:J.Physics A 26,L1189-L1193.
Nevir,P. 1998: Habilitationsschrift, FU Berlin,317p.

Nevir's Nambu formulation of hydro-thermodynamic basic equations

$$\rho \frac{d\vec{v}}{dt} = -\nabla p - \rho \nabla \phi - 2\vec{\Omega} \times \rho \vec{v}$$

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{v}$$

$$\rho \frac{dT}{dt} = -\frac{p}{c_v} \nabla \cdot \vec{v}$$

$$\frac{\partial \vec{v}}{\partial t} = \left\{ \vec{v}, \tilde{h}_a, \tilde{H} \right\} + \left\{ \vec{v}, \tilde{M}, \tilde{H} \right\} + \left\{ \vec{v}, \tilde{S}, \tilde{H} \right\}$$

$$\frac{\partial \rho}{\partial t} = \left\{ \rho, \tilde{M}, \tilde{H} \right\}$$

$$\frac{\partial (\rho s)}{\partial t} = \left\{ \rho s, \tilde{S}, \tilde{H} \right\}$$

Entropy $\tilde{S} = \iiint d\tau (\rho s) \longrightarrow s = c_p \ln \theta$

Mass $\tilde{M} = \iiint d\tau\, \rho$

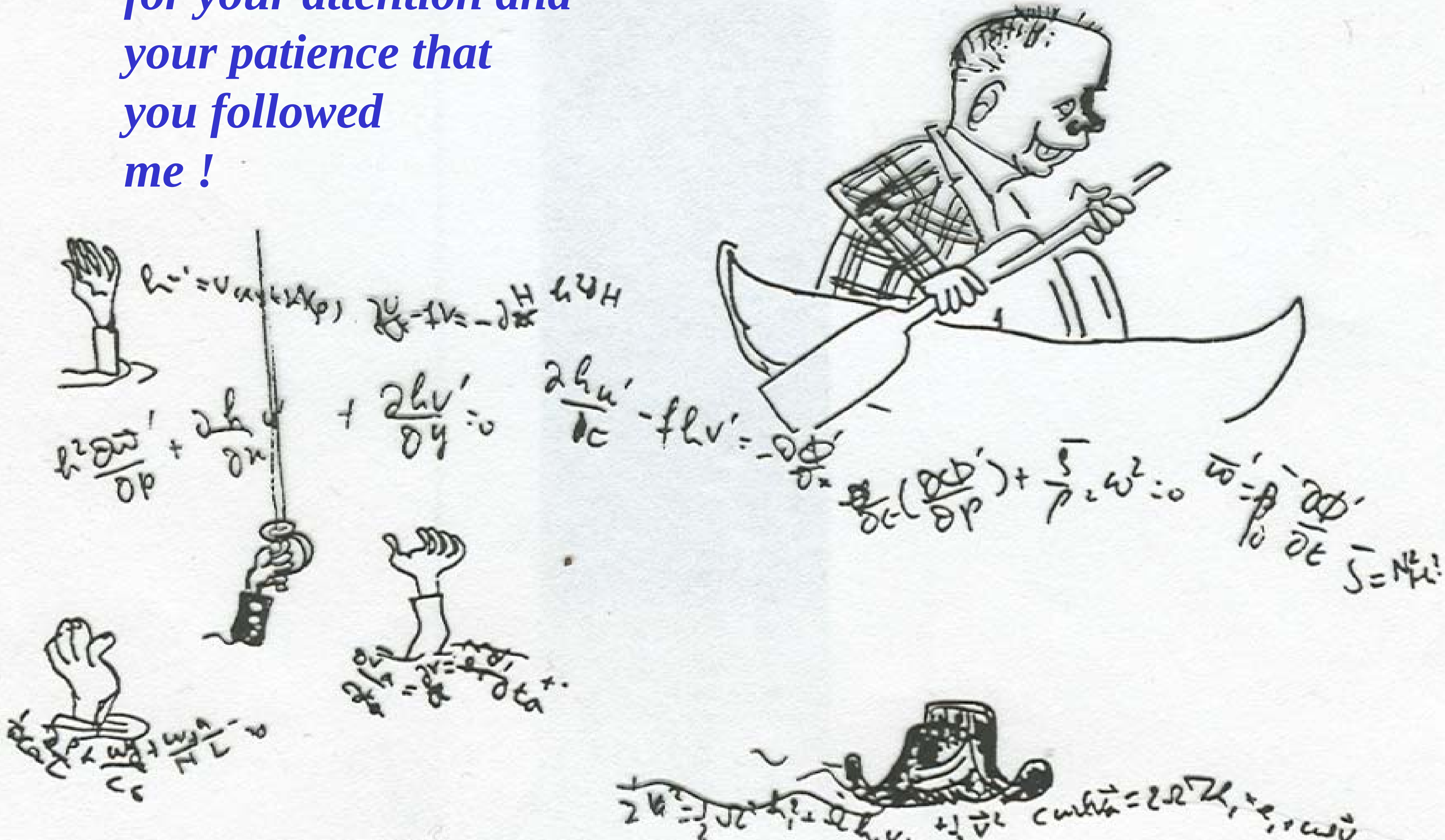
Energy $\tilde{H} = \iiint d\tau \left(\rho \vec{v}^2 / 2 + \rho \tilde{u} + \rho \phi \right)$

Helicity $\tilde{h}_a = \iiint d\tau\, \vec{v} \cdot \vec{\xi}_a \longrightarrow \vec{\xi} = \nabla_3 \times \vec{v}$

A Nambu-bracket approach to construct conserving algorithms should be possible

!

I invite you to take place in this boat !
. . . and in any case, thank you for your attention and your patience that you followed me !



Cartoon of Norman Phillips, from: Dynamic Meteorology (ed. P.Morel) 1973