Split explicit methods

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**Two common methods**

applied to nonhydrostatic compressible equations

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- **fast waves**
- **time stepping**
- **advantages**
- **difficulties**
Split explicit methods

1) How to divide the terms into slow and fast ones?

2) How to couple slow and fast processes tightly?

3) How to ensure numerical stability?

4) How to define the advection algorithm?
Divide equations into slow and fast parts – Investigate fast modes

Linear wave analysis of...

\[
\frac{\partial \hat{u}}{\partial t} + \frac{\partial \hat{p}}{\partial x} = 0
\]

\[
\frac{\partial \hat{w}}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{2H} + \frac{g}{RT} \right) \hat{p} - \frac{g}{c_pT} \hat{T} = 0
\]

\[
\frac{\partial \hat{p}}{\partial t} - g \hat{w} + c_s^2 \left( \frac{\partial \hat{u}}{\partial x} + \left( \frac{\partial}{\partial z} + \frac{1}{2H} \right) \hat{w} \right) = 0
\]

\[
\frac{\partial \hat{T}}{\partial t} - \mu g \left( 1 - \frac{N^2}{N_0^2} \right) \hat{w} + c_s^2 \left( \frac{\partial \hat{u}}{\partial x} + \left( \frac{\partial}{\partial z} + \frac{1}{2H} \right) \hat{w} \right) = 0
\]

...gives us the dispersion relation for acoustic and gravity waves...

\[
\frac{1}{c_s^2} \omega^4 - \omega^2 \left( k^2 + m^2 + \frac{1}{4H^2} \right) + k^2 N^2 = 0
\]

\[
\omega_a^2 \approx c_s^2 \left( k^2 + m^2 + \frac{1}{4H^2} \right)
\]

\[
\omega_g^2 \approx \frac{k^2 N^2}{k^2 + m^2 + \frac{1}{4H^2}}
\]

...wherein these both modes are coupled via the divergence.

As a consequence the acoustic and gravity modes are not strictly separable. But this was ignored in the past by the pioneers in this working field.
Numerical stability analysis of (LM–equations) ...

$$\frac{u^{\nu+1} - u^{\nu}}{\Delta \tau} = -\frac{\partial p^{\nu}}{\partial x} + \alpha_h \Delta \tau c_s^2 \frac{\partial D^\nu}{\partial x}$$

terms for divergence damping

$$\frac{w^{\nu+1} - w^{\nu}}{\Delta \tau} = -\left( \frac{\partial}{\partial z} - \frac{1}{2H} \right) \left( \beta^+ p^{\nu+1} + \beta^- p^{\nu} \right) -$$

$$-\frac{g}{RT} \left( \beta^+ p^{\nu+1} + \beta^- p^{\nu} \right) + \frac{g}{c_p T} \left( \beta^+_{l m} T^{\nu+1} + \beta^-_{l m} T^{\nu} \right)$$

$$+ \alpha_v \Delta \tau c_s^2 \left( \frac{\partial}{\partial z} + \frac{1}{2H} \right) \left( \beta^+ D^{\nu+1} + \beta^- D^{\nu} \right)$$

critical points concerning LM / MM5

$$\frac{T^{\nu+1} - T^{\nu}}{\Delta \tau} = -c_s^2 \left( \frac{\partial}{\partial z} + \frac{1}{2H} \right) \left( \beta^+ w^{\nu+1} + \beta^- w^{\nu} \right) - c_s^2 \frac{\partial u^{\nu+1}}{\partial x} +$$

$$+ \mu g \left( 1 - \frac{N^2}{N_0^2} \right) \left( \beta^+ w^{\nu+1} + \beta^- w^{\nu} \right)$$

$$\frac{p^{\nu+1} - p^{\nu}}{\Delta \tau} = \left( g - c_s^2 \left( \frac{\partial}{\partial z} + \frac{1}{2H} \right) \right) \left( \beta^+ w^{\nu+1} + \beta^- w^{\nu} \right) - c_s^2 \frac{\partial u^{\nu+1}}{\partial x}$$
Divide equations into slow and fast parts – Investigate fast modes

...yields a matrix equation like

\[ B\Phi^{\nu+1} = C\Phi^\nu \]

...and an amplification matrix

\[ A = B^{-1}C \]

...whose eigenvalues must not exceed 1 for numerical stability.

But...

If the temperature part of the buoyancy term are treated explicitly for an isothermal atmosphere, the gravity waves become unstable!
For the correct representation of the actual stability ($N^2$), the vertical advection of pressure and temperature is required in the fast waves part.

$$N^2 = 0.0001/s^2$$

Dropping vertical advection of $T$ with nonisothermal reference state.

(LM pathology)

Vertical advection of $T$ and $p'$ is performed with the nonisothermal reference state.
As we shall see, some damping of sound waves is required in the splitting scheme...

Divergence damping in the horizontal momentum equations

Divergence damping in all three momentum equations

Off-centered implicit weights ($\beta_+=0.7$)

Divide equations into slow and fast parts – Investigate fast modes
Divergence damping analysis

**Isotropic divergence damping**

\[
\frac{1}{c_s^2} \omega^4 + i\gamma \left( k^2 - \left( im + \frac{1}{2H} \right)^2 \right) \frac{\omega^3}{c_s^2} - \kappa^2 \omega^2 + k^2 N^2 = 0
\]

damping coefficient

\[
\omega_s^{+, -} \approx \pm \kappa c_s \left( 1 - \frac{\gamma^2 k^2}{4c_s^2} \right)^{1/2} - i\frac{1}{2} \gamma k^2
\]

Sound waves are damped. Gravity waves remain unmodified.

\[
\omega_g^{+, -} \approx \pm \frac{kN}{\kappa}
\]

**Horizontal divergence damping**

\[
\frac{1}{c_s^2} \omega^4 + i\gamma_h k^2 \frac{\omega^3}{c_s^2} - \kappa^2 \omega^2 - i\gamma_h k^2 \left( im + \frac{1}{2H} \right) \frac{g}{c_s^2} \omega + k^2 N^2 = 0
\]

\[
\omega_s^{+, -} \approx \pm \kappa c_s \left( 1 - \frac{\gamma_h^2 k^4}{4c_s^2 \kappa^2} \right)^{1/2} - i\frac{1}{2} \gamma_h k^2
\]

Sound waves are damped proportionally to the horizontal wave number.

\[
\omega_g^{+, -} \approx \pm \frac{kN}{\kappa} \left( 1 + \frac{1}{4} \gamma_h \frac{k^2}{k^2 N^2} \left( \frac{g}{c_s^2 \kappa^2} \right)^2 \right)^{1/2} + \frac{1}{2} \gamma_h \frac{g k^2 m}{c_s^2 \kappa^2}
\]

Gravity waves are altered in phase and become faster or slower.

Divide equations into slow and fast parts – Investigate fast modes.
Conclusions from this section

• Numerical stability is required for fast waves part alone.

• A horizontally forward–backward explicit and vertically implicit numerical scheme is applied.

• Acoustic and gravity modes are coupled via the divergence and, therefore, are not separable.

• All terms relevant for vertical structure and wave propagation must be treated within the fast waves part and with the same implicity weights.

• Divergence damping should only be used if it is applied to all three momentum equations. Off-centering the implicity weights is an alternative damping mechanism.
Numerical analysis of the sound advection system

\[ \frac{\partial u}{\partial t} + c_s \frac{\partial \hat{p}}{\partial x} = -U \frac{\partial u}{\partial x} \]
\[ \frac{\partial \hat{p}}{\partial t} + c_s \frac{\partial u}{\partial x} = -U \frac{\partial \hat{p}}{\partial x} \]

fast modes
sound waves
slow modes
advection terms

For comparison of different schemes we must define a common advection scheme:
Runge–Kutta 2\textsuperscript{nd} order in time and 3\textsuperscript{rd} order in space.

Klemp/Wilhelmson
Euler forward
Wicker/Skamarock
Gassmann
Numerical analysis of the sound advection system

Example: Euler scheme

\[
\begin{align*}
    u_{i+1/2}^{m+1} &= u_{i+1/2}^m - \Delta \tau c_s \frac{\hat{p}_{i+1}^m - \hat{p}_i^m}{\Delta x} + \\
    \Delta \tau G(u)^n_{i+1/2} \\
    \hat{p}_i^{m+1} &= \hat{p}_i^m - \Delta \tau c_s \frac{u_{i+1/2}^{m+1} - u_{i-1/2}^{m+1}}{\Delta x} + \\
    \Delta \tau G(\hat{p})^n_i
\end{align*}
\]

RK2–Advection
(Runge–Kutta 2\(^{nd}\) order in time)

\[
\begin{align*}
    \psi_i^{m+1} &= \psi_i^m + \frac{\Delta t}{2} F(\psi^n_i) \\
    \psi_i^{n+1} &= \psi_i^n + \Delta t F(\psi_i^{n*}) \\
    G(\psi)^n_i &= \frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} = F(\psi_i^{n*}) = F(\psi^n + \frac{\Delta t}{2} F(\psi^n_i))
\end{align*}
\]

3\(^{rd}\) order in space (for \(U>0\))

\[
F(\psi)_i = -\frac{U}{6\Delta x} (\psi_{i-2} - 6\psi_{i-1} + 3\psi_i + 2\psi_{i+1})
\]

Fourier representation in space

\[
\tilde{k}_U = \frac{1}{3} (\sin(k\Delta x) (4 - \cos(k\Delta x))
\]

\[-i(\cos(k\Delta x) - 1)^2).
\]

Imaginary part is negative and leads to damping
Numerical analysis of the sound advection system

The computations are performed on a staggered C-grid with forward-backward differencing for the fast waves part and the commonly defined advection algorithm.

Number of small time steps per large step: $N=4$
$u=0.75, cs=3, dx=dt=1$ are nondimensional numbers

First 8 time steps

Wicker/Skamarock

Euler forward

Gassmann
Numerical analysis of the sound advection system

Expansion of the squared eigenvalues of the amplification matrix:

\[ |\lambda_{1,2}|^2 = 1 + 2\Im(k_U)U + 2\Im(k_U)^2U^2 \pm S.E.T. + H.O.T. \]

- Wavenumber for the advection scheme
- Real part: phase characteristic
- Imaginary part (is negative): damping

Splitting error term

Higher order terms

Courant numbers

Euler forward

\[ S.E.T. = Nk_{cs} c_s \Re(k_U)U \]

Wicker/Skamarock

\[ S.E.T. = 0.625 Nk_{cs} c_s \Re(k_U)\Im(k_U)U^2 \]

Gassmann

\[ S.E.T. = 0.125 Nk_{cs} c_s \Re(k_U)\Im(k_U)U^2 \]
Numerical analysis of the sound advection system

Stability diagrams for an $8 \Delta x$ wave

**Euler forward**

Forward moving mode

with divergence damping

**Wicker/Skamarock**

**Gassmann**

Backward moving mode
Splitting scheme analysis with the linear nonhydrostatic compressible system

Stability diagrams of the Gassmann scheme

- no divergence damping, no off-centering of implicit weights
  - unstable

- isotropic divergence damping, no off-centering of implicit weights
  - stable

- no divergence damping, off-centering of implicit weights
  - stable
Splitting schemes – Conclusions

• Though the advection scheme and the fast–waves scheme may be stable for themselves, the combination in the splitting scheme is not automatically stable!

• The splitting error term is a multiplicative combination of both parts and contains the significant propagation information, and so never vanishes: it may only be reduced.

• An additional damping mechanism (hidden in H.O.T.) is essential.

• The propagation of waves in different directions (modes) is either amplified or damped.

• The Gassmann–scheme is shown to be the best compromise among the candidates presented.

• The stability analysis of the splitting scheme for the complete nonhydrostatic compressible equations yields satisfactory results.
A gravity wave generator is situated in the center of the domain, the ambient horizontal wind increases with height from 5 to 15 m/s.

Since the operators for each fractional step do not commute, the stability of each individual operator no longer guarantees the stability of the overall scheme.

From Durran (1999)
Other variants of splitting schemes – Runge Kutta type advection

![Diagram showing Runge-Kutta-2nd and 3rd order in time]

**Table 1. Maximum stable Courant number for one-dimensional linear advection.** Here, U indicates the scheme is unstable.

<table>
<thead>
<tr>
<th>Time scheme</th>
<th>Spatial order</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leapfrog</td>
<td></td>
<td>U</td>
<td>0.72</td>
<td>U</td>
<td>0.62</td>
</tr>
<tr>
<td>RK2</td>
<td></td>
<td>0.88</td>
<td>U</td>
<td>0.30</td>
<td>U</td>
</tr>
<tr>
<td>RK3</td>
<td></td>
<td>1.61</td>
<td>1.26</td>
<td>1.42</td>
<td>1.08</td>
</tr>
</tbody>
</table>

From Wicker and Skamarock (2002), cf. also WRF–Documentation

But the Gassmann scheme is independent of the actual advection scheme and may be combined with RK3.

Larger Courant numbers for RK3 and higher accuracy in space!
Applications – Consequences for the LM

The nonhydrostatic compressible LM (Lokal–Modell) is the operational regional forecast model of the COSMO group. Its dynamical core reads:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{\rho a \cos \varphi} \left( \frac{\partial p'}{\partial \lambda} + \frac{1}{\sqrt{G}} \frac{\partial z}{\partial \lambda} \frac{\partial p'}{\partial \zeta} \right) &= s_u, & s_u &= f v - \frac{1}{a \cos \varphi} \left( u \frac{\partial u}{\partial \lambda} + v \cos \varphi \frac{\partial u}{\partial \varphi} \right) - \xi \frac{\partial u}{\partial \zeta} + \frac{uv}{a} \tan \varphi \\
\frac{\partial v}{\partial t} + \frac{1}{\rho a} \left( \frac{\partial p'}{\partial \varphi} + \frac{1}{\sqrt{G}} \frac{\partial z}{\partial \varphi} \frac{\partial p'}{\partial \zeta} \right) &= s_v, & s_v &= -f u - \frac{1}{a \cos \varphi} \left( u \frac{\partial v}{\partial \lambda} + v \cos \varphi \frac{\partial v}{\partial \varphi} \right) - \xi \frac{\partial v}{\partial \zeta} - \frac{u^2}{a} \tan \varphi \\
\frac{\partial w}{\partial t} - \frac{1}{\rho \sqrt{G}} \frac{\partial p'}{\partial \zeta} + \frac{g}{\rho T R_d} p' - g \frac{\rho_0}{\rho} \left( \frac{\partial p}{\partial \zeta} - \frac{T_0}{T} \right) &= s_w, & s_w &= -\frac{1}{a \cos \varphi} \left( u \frac{\partial w}{\partial \lambda} + v \cos \varphi \frac{\partial w}{\partial \varphi} \right) - \xi \frac{\partial w}{\partial \zeta} \\
\frac{\partial p'}{\partial t} - \frac{1}{\sqrt{G}} \frac{\partial p}{\partial \zeta} w + \frac{c_{pd}}{c_{vd}} p' \left( \frac{\partial w}{\partial \zeta} - \frac{1}{\sqrt{G}} \frac{\partial w}{\partial \zeta} \right) &= s_{p'}, & s_{p'} &= -\frac{1}{a \cos \varphi} \left( u \frac{\partial p'}{\partial \lambda} + v \cos \varphi \frac{\partial p'}{\partial \varphi} \right) - \xi \frac{\partial p'}{\partial \zeta} \\
\frac{\partial T}{\partial t} - \frac{1}{\sqrt{G}} \frac{\partial T}{\partial \zeta} w + \frac{1}{c_{vd} \rho} \left( \frac{\partial w}{\partial \zeta} - \frac{1}{\sqrt{G}} \frac{\partial w}{\partial \zeta} \right) &= s_T, & s_T &= -\frac{1}{a \cos \varphi} \left( u \frac{\partial T}{\partial \lambda} + v \cos \varphi \frac{\partial T}{\partial \varphi} \right) - \xi \frac{\partial T}{\partial \zeta} \\
\dot{\zeta} &= \dot{\xi} - \frac{1}{\sqrt{G}} w \\
\text{usual contravariant vertical velocity} \\
\text{vertical velocity related to the terrain following coordinates}
\end{align*}
\]
Applications – Consequences for the LM

Schaer test case with Gassmann-splitting

vertical advection of $T$ and $p'$ in slow modes  
vertical advection of $T$ and $p'$ in fast modes
With the new fast waves algorithm all information for gravity waves is included in the fast-waves part. That is the prerequisite for applying the radiative upper boundary condition directly in the vertical implicit solver of the fast-waves.

\[ \hat{w} = \frac{K}{N_0} \hat{\rho}' \]

relation for hydrostatic gravity waves at the model top.
Applications – Consequences for the LM

Schaer test case with RUBC
Applications – Consequences for the LM

Schaer test case with sponge upper boundary condition
Applications – Consequences for the LM

Schaer test case with sponge upper boundary condition

old LM dynamics
Nonlinear flow past a high mountain, \( dx = 7 \text{km} \), tropopause at 10km, realistic vertical levels
Applications – Consequences for the LM

New LM dynamics. With sponge layer and with RUBC.

Old LM dynamics. Without and with dynamic lower boundary condition in the horizontal momentum equations.

Resting atmosphere over a high mountain.
Applications – Consequences for the LMK (LM–short range forecast)

TVD–Runge–Kutta scheme

\[
\psi^n_i = \psi^n_i + \Delta t F(\psi^n_i), \\
\psi^{**}_i = \frac{3}{4} \psi^n_i + \frac{1}{4} \psi^*_i + \frac{1}{4} \Delta t F(\psi^*_i), \\
\psi^{n+1}_i = \frac{1}{3} \psi^n_i + \frac{2}{3} \psi^{**}_i + \frac{2}{3} \Delta t F(\psi^{**}_i).
\]

**Solid Body Rotation** of a tracer cone with an initial maximum of 1, 400 time steps for one turn

RK–3rd / CD–4th  
RK–3rd / CD–4th + HD  
TVD–RK–3rd / CD–4th

670 time steps  
550 time steps  
450 time steps

Figures from Jochen Foerstner, DWD
Applications – Consequences for the LMK (LM–short range forecast)

TVD–Runge–Kutta scheme now applied within the framework of the Wicker/Skamarock–splitting

TVD–RK3/4\textsuperscript{th}CD + horizontal diffusion

advection of a tracer without fast–waves

TVD–RK3/5\textsuperscript{th} upwind
Applications – Conservative split–explicit WRF version

WRF – Weather Research and Forecasting modeling system
collaboration amongst NCAR, NOAA, FSL, AFWA, NRL, CAPS, FAA in the U.S.A.

Flux quantities
\[ \mathbf{V} = \rho \mathbf{v} = (U, V, W), \quad \Theta = \rho \theta, \]

Flux form equations
\[
\begin{align*}
\partial_t U + \nabla \cdot (\mathbf{v} U) + \partial_x p' & = F_U \\
\partial_t V + \nabla \cdot (\mathbf{v} V) + \partial_y p' & = F_V \\
\partial_t W + \nabla \cdot (\mathbf{v} W) + \partial_z p' + g \rho' & = F_W \\
\partial_t \Theta + \nabla \cdot (\mathbf{v} \Theta) & = F_\Theta \\
\partial_t \rho' + \nabla \cdot \mathbf{V} & = 0.
\end{align*}
\]

How to linearize these equations for splitting off the fast–waves part?
\[
\mathbf{V}'' = \mathbf{V} - \mathbf{V}^t = (U - U^t, V - V^t, W - W^t), \quad \Theta'' = \Theta - \Theta^t, \text{ and } \rho'' = \rho - \rho^t
\]

This corresponds to a linearization around the present time step \( t \).

From Klemp et al., 2000, cf. also Skamarock et al. 2005

\[ \partial_t U'' + \gamma R \pi^t \partial_x \Theta'' = F_{U^t} - \partial_x p''^t - \nabla \cdot (\mathbf{v}^t U^t) \]
\[ \partial_t V'' + \gamma R \pi^t \partial_y \Theta'' = F_{V^t} - \partial_y p''^t - \nabla \cdot (\mathbf{v}^t V^t) \]
\[ \partial_t W'' + \gamma R \pi^t \partial_z \Theta'' - g \rho'' \frac{R}{c_v} \frac{\Theta''}{\pi} + g \rho'' = F_{W^t} - \partial_z p''^t - g \rho''^t - \nabla \cdot (\mathbf{v}^t W^t) \]
\[ \partial_t \Theta'' + \nabla \cdot (\mathbf{V}'' \Theta^t) = F_{\Theta^t} - \nabla \cdot (\mathbf{v}^t \Theta^t) \]
\[ \partial_t \rho'' + \nabla \cdot \mathbf{V}'' = -\nabla \cdot \mathbf{V}^t \]
Summary on split–explicit methods

• The split explicit method is an efficient and accurate method for integrating the unfiltered hydro–thermodynamic equations.

• The method is easily implemented also on parallel platforms.

• Numerical stability is the crucial point in designing split–explicit schemes.

• Another problem is the proper mode splitting.

• The combination with different advection schemes is possible.

• Split–explicit time integration is even applicable to the flux–form equations.

• Features like the radiative upper boundary condition are easily included in the complete algorithm.