Spectral Formulation of NH Model

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INTRODUCTION

- Source terms have spatial derivatives (pressure force, orographic forcing, ...)
- Several possibilities :

  Low order : Finite Differences (FD) (basis : Dirac)

  Medium order : Finite Elements (FE) (basis : Sawtooth, spline,...)

  High order : spectral method (SP) (basis : harmonic function)

Application in NWP models

- Along vertical : FD (or FE)
- Along the horizontal : FD or FE or SP

Outline

Field decomposition

Along horizontal (e.g. x), any field Ψ is described as :

$$\Psi(x) = \sum_{j=-M}^{j=M} \widetilde{\psi}_j . \exp(ijx)$$

with $\widetilde{\psi_{\pm j}}$ being complex conjugate numbers
M is the truncation (number of degree of freedom)
The x-derivatives of Ψ writes :

$$\frac{\partial \Psi(x)}{\partial x} = \sum_{j=-M}^{j=M} ij\widetilde{\psi}_j . \exp(ijx)$$

and Laplacian operator :

$$\frac{\partial^2 \Psi(x)}{\partial x^2} = \sum_{j=-M}^{j=M} -j^2\widetilde{\psi}_j . \exp(ijx)$$

Transform method

$$\Psi(x) = \sum_{j=-M}^{j=M} \widetilde{\psi}_j . \exp(ijx)$$

- Any field Ψ is therefore described by its "Fourier complex spectrum" $\widetilde{\psi}_j$ instead of its values on a stencil
- But if X and Y are known fields, how to describe the combination of terms like $Y.Z$ ot $\exp(Y)$ or even more complicated ... ?
- For terms like $Y.Z$ we could combine spectra directly , obtaining a spectrum $[-2M, 2M]$,and truncate to M to obtain the description of the product.
- But this method cannot be applied for general operators
- Instead : we use the so-called "transform method"

Transform method

Consider e.g. :

$$\frac{\partial X(x)}{\partial t} = \nabla(Y.Z) \quad \text{and} \quad (Y,Z) = f(X)$$

- $\widetilde{X_j(t)}$ is known (in spectral form) ;
- FFT^{-1} gives $X_k(t)$ in physical space on a proper stencil x_k ;
- compute $Y_k(t)$ and $Z_k(t)$ by applying f operator ;
- compute $YZ_k(t) = Y_k(t).Z_k(t)$ in physical space ;
- FFT gives $\widetilde{YZ_j(t)}$ in spectral space (spectral representation of YZ) ;
- compute $\nabla(YZ)$ spectrally ;
- perform time-marching scheme in spectral space, giving $\widetilde{X_j(t+1)}$
- ...

Transform method

How to choose the physical stencil x_k ?

- stencil x_k with $k \in [1, K]$;
- K resolution in physical space $\leftrightarrow$ M spectral truncation (resolution) ;
- For aliasing-free quadratic products we would need $K \approx 3M$;
- But for Semi-Lagrangian models quadratic terms (advective) are no longer dominant
- $\Rightarrow$ in practice we choose $K \approx 2M$

Periodicisation needed

- For Fourier formalism, peropdicity needed
- Achieved by biperiodicisation in artificial extension area (cubic splines)
- Slightly more computations

Outline

Bad : the Gibbs problem for transform method

- Around physical discontinuities (cliffs, clouds, ?), the physical description of a spectrally truncated field contains somehow "unphysical" features : Gibbs osillations.
- Becomes "not smaller" when resolution increases (see example)
- Potentially leads to e.g. negative sea-level height near cliff
- Potentially lead to negative moisture content near cloud edge
- $\Rightarrow$ needs specific fix for these drawbacks

Good : the nice Laplacian operator

- Laplacian operator is VERY central to models for implicit schemes, diffusion,...
- Laplacian operator needs to be inverted for implicit schemes
- In spectral space Laplacian is diagonal (trivial to inverse) !
- The FFT can be viewed as a DIRECT solver for the Laplacian operator
- $\Rightarrow$ leads to potentially very efficient models (IFS)
- But warning, it does not allow to invert terms like $X\nabla^2 Y$
- $\Rightarrow$ limited advantage for highly non-linear systems
  (as e.g. High Resolution compressible models)
- maybe not well adapted for the future H.Resol NWP models ?

Outline

Spectral Formulation of the NH model

- If the NH model is explicit, spectral formulation does not cause any problem : just apply the abovementionned transform method in a straightforward way.
- But due to very fast acoustic waves in compressible models, it's necessary to formulate the evolution in an implicit way, for efficiency ;
- This leads to inversion of Laplacian operator (Helmholtz equation)
- Spectral formulation is then central to the (implicit) model design

Spectral Formulation of the NH model

e.g. in pure σ coordinate :

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} + RT\frac{\boldsymbol{\nabla} p}{p} + \frac{1}{\pi_s}\frac{\partial p}{\partial \sigma}\boldsymbol{\nabla}\phi &= 0 \\
\frac{dw}{dt} + g\left(1 - \frac{1}{\pi_s}\frac{\partial p}{\partial \sigma}\right) &= 0 \\
\frac{dT}{dt} - \frac{RT}{C_v}.(\boldsymbol{\nabla}_3.\mathbf{V}) &= 0 \\
\frac{dp}{dt} + \frac{C_p}{C_v}p(\boldsymbol{\nabla}_3.\mathbf{V}) &= 0 \\
\frac{\partial q}{\partial t} + \int_0^1 (\mathbf{V}\boldsymbol{\nabla} q + \boldsymbol{\nabla}\mathbf{V})d\sigma &= 0
\end{aligned}
$$

where $q = \ln(\pi_s)$ and $(\boldsymbol{\nabla}_3.\mathbf{V})$ is the true 3D divergence
Notice various non-linear gradient terms

Spectral Formulation of the NH model

Remember :for designing SI (or ICI) scheme, we must define a linear system $\mathbf{L}^*$: we choose a reference state X^* and linearize around X^* : $[\mathbf{V}^* = w^* = 0, T^* = \mathrm{Cst},\ p^* = \pi^*(\sigma), \phi^*(\sigma), q^* = \mathrm{Cst}]$

$$
\begin{aligned}
\frac{d\mathbf{V}'}{dt} + RT^* \frac{\boldsymbol{\nabla} p'}{\pi *} + \boldsymbol{\nabla}\phi' &= 0 \\
\frac{dw'}{dt} - g\sigma \frac{\partial p'}{\partial \sigma} &= 0 \\
\frac{dT'}{dt} - \frac{RT^*}{C_v}.(\boldsymbol{\nabla}_3.\mathbf{V}') &= 0 \\
\frac{dp'}{dt} + \frac{C_p}{C_v}\pi^*(\boldsymbol{\nabla}_3.\mathbf{V}') &= 0 \\
\frac{\partial q'}{\partial t} + \int_0^1 (\boldsymbol{\nabla}\mathbf{V}')\, d\sigma &= 0
\end{aligned}
$$

Spectral Formulation of the NH model

The linear terms are treated implicitly while the non-linear residuals are treated explicitly, as described yesterday :

$$
\begin{aligned}
\frac{\delta \mathbf{V}}{\delta t} &= (\mathbf{N}_V - \mathbf{L}_V^*)(t) + [\mathbf{L}_V^*(t+1) + \mathbf{L}_V^*(t-1)]/2 \\
\frac{\delta w}{\delta t} &= (\mathbf{N}_w - \mathbf{L}_w^*)(t) + [\mathbf{L}_w^*(t+1) + \mathbf{L}_w^*(t-1)]/2 \\
\frac{\delta T}{\delta t} &= (\mathbf{N}_T - \mathbf{L}_T^*)(t) + [\mathbf{L}_T^*(t+1) + \mathbf{L}_T^*(t-1)]/2 \\
\frac{\delta p}{\delta t} &= (\mathbf{N}_p - \mathbf{L}_p^*)(t) + [\mathbf{L}_p^*(t+1) + \mathbf{L}_p^*(t-1)]/2 \\
\frac{\delta q}{\delta t} &= (\mathbf{N}_q - \mathbf{L}_q^*)(t) + [\mathbf{L}_q^*(t+1) + \mathbf{L}_q^*(t-1)]/2
\end{aligned}
$$

Spectral Formulation of the NH model

$$\frac{\delta \mathbf{V}}{\delta t} = (\mathbf{N}_V - \mathbf{L}^*_V)(t) + [\mathbf{L}^*_V(t+1) + \mathbf{L}^*_V(t-1)]/2$$

$$\vdots$$

$$\frac{\delta q}{\delta t} = (\mathbf{N}_q - \mathbf{L}^*_q)(t) + [\mathbf{L}^*_q(t+1) + \mathbf{L}^*_q(t-1)]/2$$

- The system in [V(t+1),w(t+1),T(t+1),p(t+1),q(t+1)] is closed
- All coefficients of spatial operators are horizontally constant
- All horizontal operators commute
- All variables but one can be algebraically eliminated
- $\longrightarrow$ single Helmholtz equation for a single variable

Spectral Formulation of the NH model

- Like in any SI model, the details of the algebraic elimination are quite cumbersome (but rather automatic)
- Finally, the Helmholtz equation looks like :

$$\left[1 - \delta t^2 c_*^2 \left(\mathbf{m}_*^2 \Delta' + \frac{\mathbf{L}_v^*}{r H_*^2}\right) - \delta t^4 \frac{N_*^2 c_*^2}{r} \mathbf{m}_*^2 \Delta' \mathbf{T}^*\right] \underline{\mathrm{d}}(t+1) = \underline{\mathrm{d}}^{\bullet\bullet}$$

where :

- $\underline{\mathrm{d}}(t+1)$ is the unknown prognostic variable at time $(t+1)$
- $\mathbf{L}_v^*$ and $\mathbf{T}^*$ are vertical discrete operators
- $\Delta' = \nabla^2$ is the horizontal Laplacian operator (to be inversed)
- $\underline{\mathrm{d}}^{\bullet\bullet}$ is the so-called "RHS" containing only known information (from times t and $t-1$)

Spectral Formulation of the NH model

$$\left[1 - \delta t^2 c_*^2 \left(\mathbf{m}_*^2 \Delta' + \frac{\mathbf{L}_v^*}{rH_*^2}\right) - \delta t^4 \frac{N_*^2 c_*^2}{r} \mathbf{m}_*^2 \Delta' \mathbf{T}^*\right] \underline{\mathrm{d}}(t+1) = \underline{\mathrm{d}}^{\bullet\bullet}$$

- The Helmholtz equation is trivially solved in spectral space and in the vertical eigenmodes space :
- The inversion of the LHS operator can even be done once at the begining of the forecast, for each horizontal and vertical eigenmode of this operator
- This leads to a very efficient formulation

Spectral Formulation of the NH model

$$(1 - .5\Delta t\mathbf{L})X_A^+ = \Delta t(\mathbf{M} - \mathbf{L})(X_M^0) + (1 + .5\Delta t\mathbf{L})X_D^-(= RHS)$$

$$X_A^+ = (1 - .5\Delta t\mathbf{L})^{-1}.RHS$$

Outline of a model time-step

- Begin with X_A^+ in spectral space
- Transfer it to GP space by FFT^{-1} (and relabel it as X^0)
- Compute all dynamical terms and physical tendencies on the grid ($\mathbf{M}.X^0, \mathbf{L}.X^0, \mathbf{L}.X^-$)
- Perform SL computations (origin points, interpolations...) ($\mathbf{M}.X_M^0, \mathbf{L}.X_D^-,\ldots$)
- Gather all in RHS and transfer to SP space through FFT
- Solve the linear implicit operator $\rightarrow X_A^+$

Outline

What Future for Spectral Formulation in NH LAMs ?

- Such a strategy for the SI model leads to very efficient formulation
- However it allow an implicit treatment only for a small part of the flow

For instance :

$$\frac{\partial q}{\partial t} + \int_0^1 (\mathbf{V}\boldsymbol{\nabla} q + \boldsymbol{\nabla}\mathbf{V}) d\sigma = 0$$

Near surface : big orographic term $\mathbf{V}.\boldsymbol{\nabla} q \approx \mathbf{V}_s.\boldsymbol{\nabla}\phi_s/(RT_s)$

- Stability of SI (or convergence of ICI) is not guaranteed
- This very efficient spectral formulation could need to be revised when approaching 100-500 m scales in NWP, because of too large nonlinearity of the system.

What Future for Spectral Formulation in NH LAMs ?

- Other domains where spectral method is constraining for NH LAMs :
- Tansparent LBCs (Mc Donald) very hard (impossible ?) in spectral models (however Perfectly Matched Layers (PML) are very promising)
- Two-way nesting seems impossible in spectral formalism
- ... ?

Personal statement :

- Spectral technique will probably have to be abandoned for NH LAM when approaching $\Delta x \approx 500 - 100m$
- Our community should prepare to this change
- We must get familiar with critical/difficult points of FD schemes
- These are not necessarily well documented due to research competition !

Spectral technique was not a bad idea (in 80'-90's)

What Future for Spectral Formulation in NH LAMs ?

An example of concealed feature

- MC2 is a Canadian NH LAM (FD) model with reputation of being very clean since more than 15 years.
- In a 2003 paper, I did an analysis predicting that MC2, as it is documented in papers, should be unstable, unless a significant decentering $\epsilon = 0.05$ is used (with detrimental effect on accuracy).
- The cause of the instability was also explained by the analysis.
- In MWR, 2005, Girard et al. writes :

Girard et al., MWR, 2005

" ... after Bénard (2003) evaluation of our SI scheme showing that in the absence of both a time-filter and off-centering the scheme was actually absolutely unstable (and in fact without off-centering the model is known to blow up sometimes), we have developed a more stable SI scheme [...] which does not require off-centering... "

What Future for Spectral Formulation in NH LAMs ?

- Examples like this are rather common
- This means that research teams tend to minimize their difficulties in publications.
- Consequently, it is difficult to honnestly say if migration from spectral NH to FD NH in Aladin will be a hard task or not !
- Effort should be progressively devoted to this task.
- The main points are the matrix inverse solver and the derivation of the SI scheme
- Of course this Finite-Difference version should be implemented as a optional feature.

Outline

Conclusions

- Spectral method allows an acurate discretization along horizontal
- A transform method must be used (for SL and non-linear computations)
- Spectral method can be used for Euler Equations NWP at km scales
- The techniques needed are the same as for HPEs
- This leads to robust and efficient models at km scales
- For 100 m scales, the robustness could suffer (steep slopes)
- Moreover, spectral method makes special features difficult (especially sophisticated LBCs and coupling)
- Strategically, a FD version should be considered and prepared