B Reverted bubble test

Aim of this section is to show that when Boussinesq approximation is applicable, reverted bubble test should give the same results as direct test.

Dynamical equations describing irrotational adiabatic frictionless atmosphere composed of perfect gas are usually written in the form:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{RT}{p}\nabla p - \nabla\phi \tag{33}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\varkappa p \,\nabla\cdot\mathbf{v} \tag{34}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -(\varkappa - 1)T\,\nabla\cdot\mathbf{v} \tag{35}$$

$$\phi \equiv gz \qquad \varkappa \equiv \frac{c_p}{c_v}$$

Standard notations are used: $\mathbf{v}$ is 3D velocity with components (u, v, w), p is pressure, T is thermodynamical temperature, ϕ is geopotential, g is gravity acceleration, R is gas constant of dry air, c_p and c_v are specific heats of dry air at constant pressure and at constant volume.

Equations (33)–(35) can be rewritten into more suitable form using non-dimensional Exner function Π and potential temperature θ:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -c_p\theta\,\nabla\Pi - \nabla\phi \tag{36}$$

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = -(\varkappa - 1)\Pi\,\nabla\cdot\mathbf{v} \tag{37}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0 \tag{38}$$

$$\Pi \equiv \left(\frac{p}{p_{00}}\right)^{\kappa} \qquad \theta \equiv T\left(\frac{p_{00}}{p}\right)^{\kappa}$$

$$\kappa \equiv \frac{R}{c_p} \qquad p_{00} \equiv 1000\,\mathrm{hPa}$$

Quantities Π and θ can be decomposed into background values and perturbations:

$$\Pi = \Pi_0 + \Pi' \qquad \theta = \theta_0 + \theta' \tag{39}$$

Background state is chosen resting, hydrostatically balanced and neutrally stratified (isoentropic). This gives:

$$\Pi_0(z) = \Pi_0(0) - \frac{gz}{c_p\theta_0} \qquad \theta_0 = const \tag{40}$$

Inserting (39) and (40) into system (36)–(38) with restriction to xz plane leads to:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -c_p(\theta_0 + \theta')\frac{\partial \Pi'}{\partial x} \tag{41}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -c_p(\theta_0 + \theta')\frac{\partial \Pi'}{\partial z} + g\frac{\theta'}{\theta_0} \tag{42}$$

$$\frac{\partial \Pi'}{\partial t} + u\frac{\partial \Pi'}{\partial x} + w\frac{\partial \Pi'}{\partial z} = \frac{gw}{c_p\theta_0} - (\varkappa - 1)(\Pi_0 + \Pi')\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \tag{43}$$

$$\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z} = 0 \tag{44}$$

It should be mentioned here that equations (41)–(44) still describe the full 2D system, i.e. no simplifications were used during their derivation. At this point Boussinesq approximation can be introduced. It is based on two basic assumptions:

1. Perturbation θ' is small compared to θ_0 and can be neglected in equations (41), (42) except from buoyant term $g\dfrac{\theta'}{\theta_0}$.
2. Flow is close to incompressible. This requires two things: fluid velocity much smaller than speed of sound and vertical scale of motion small compared to density scale height.

Both these assumptions were fulfilled in bubble tests described in section 3, so the use of Boussinesq approximation should be justified. When it is applied to system (41)–(44), it becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -c_p\theta_0\frac{\partial \Pi'}{\partial x} \tag{45}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -c_p\theta_0\frac{\partial \Pi'}{\partial z} + g\frac{\theta'}{\theta_0} \tag{46}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{47}$$

$$\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z} = 0 \tag{48}$$

System (45)–(48) has interesting symmetry, responsible for identical behaviour of direct and reverted bubble test. It can be revealed using vertical mirroring operator M_z defined as:

$$(M_z f)(x, z, t) \equiv f(x, H - z, t)$$

Function f must be defined for $z \in [0, H]$. It can be shown easily that operator M_z is linear and has following properties:

$$\begin{aligned} M_z\partial_t &= \partial_t M_z \\ M_z\partial_x &= \partial_x M_z \\ M_z\partial_z &= -\partial_z M_z \\ M_z(f \cdot g) &= M_z f \cdot M_z g \end{aligned}$$

Using these properties it can be verified immediately that system (45)–(48) is invariant with respect to transformation:

$$\begin{pmatrix} u \\ w \\ \Pi' \\ \theta' \end{pmatrix} \mapsto \begin{pmatrix} M_z u \\ -M_z w \\ M_z \Pi' \\ -M_z \theta' \end{pmatrix}$$

This means that when fields u, w, Π', θ' are solution of the system (45)–(48), their vertical mirroring with change of sign for w and θ' produces another solution.

Remark:

For experiments described in section 3 initial state was resting ($u = 0$, $w = 0$) and vertically balanced. If it was horizontally balanced, initial perturbation Π' would be zero. Neverthless, Π' was very small initially, so the only quantity which was actually mirrored when preparing initial state for reverted test was perturbation θ'. This small inconsistency might be the reason why there is a slight difference visible when comparing figures 10 and 14. Another possible explanation is that this difference was caused by non-Boussinesq effects allowed by model dynamics.