The momentum equation in HIRLAM
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CONTENT

- **The momentum equation**
- **Molecular friction**
- **Turbulent friction**
- **The turbulent kinetic energy (TKE) equation**
- **Parameterizations in the TKE equation**
- **Kinematic surface fluxes**
- **Rotation of surface stress**

The momentum equation

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p - \rho f \vec{k} \times \vec{V} - \rho g \vec{k} + \vec{F} \qquad (1)$$

Molecular friction

$$\begin{aligned}\vec{F} &= \frac{\partial \tau_{i1}}{\partial x_i}\vec{i} + \frac{\partial \tau_{i2}}{\partial x_i}\vec{j} + \frac{\partial \tau_{i3}}{\partial x_i}\vec{k} \\ \tau_{ij} &= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\delta_{ij}\nabla\cdot\vec{V} \qquad (2)\end{aligned}$$

$i, j = 1, 2, 3;$
$x_1 = x, x_2 = y, x_3 = z, u_1 = u, u_2 = v, u_3 = w$

$$\begin{aligned}F_x = &\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) + \lambda\nabla\cdot\vec{V}\right) + \\ &\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \\ &\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) \qquad (3)\end{aligned}$$

$$
\begin{aligned}
F_y = & \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+ \\
& \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right)+\lambda\nabla\cdot\vec{V}\right)+ \\
& \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right)
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
F_z = & \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right) \\
& \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)+ \\
& \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial z}+\frac{\partial w}{\partial z}\right)+\lambda\nabla\cdot\vec{V}\right)
\end{aligned}
\tag{5}
$$

HIRLAM: μ and λ are treated as constants

$$
\begin{aligned}
F_x &= (\mu+\lambda)\frac{\partial}{\partial x}\nabla\cdot\vec{V}+\mu\nabla^2 u \\
F_y &= (\mu+\lambda)\frac{\partial}{\partial y}\nabla\cdot\vec{V}+\mu\nabla^2 v \\
F_z &= (\mu+\lambda)\frac{\partial}{\partial z}\nabla\cdot\vec{V}+\mu\nabla^2 w
\end{aligned}
\tag{6}
$$

U-momentum equation

From mass continuity

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{V} \tag{7}$$

and

$$\frac{D}{Dt}\left(\rho\vec{V}\right) = \frac{D\rho}{Dt}\cdot\vec{V} + \rho\frac{D\vec{V}}{Dt} \tag{8}$$

follows

$$\rho\frac{D\vec{V}}{Dt} = \left[\frac{D}{Dt}\left(\rho\vec{V}\right) + \rho\left(\nabla\cdot\vec{V}\right)\cdot\vec{V}\right] \tag{9}$$

The U-momentum equation in flux form becomes

$$\begin{aligned}\rho\frac{Du}{Dt} &= \left[\frac{D}{Dt}(\rho u) + \rho u\nabla\cdot\vec{V}\right] \\ &= \left[\frac{\partial}{\partial t}(\rho u) + \nabla\cdot\left(\rho u\vec{V}\right)\right] \\ &= -\frac{\partial p}{\partial x} + \rho f v + F_x \end{aligned} \tag{10}$$

Reynold decomposition

$u = \overline{u} + u'; v = \overline{v} + v'; w = \overline{w} + w'; T = \overline{T} + T'; \rho = \overline{\rho} + \rho'; p = \overline{p} + p'$

$\overline{\gamma}$ is mean and γ' is fluctuating part of variable γ governing the atmosphere.

HIRLAM:
The Boussinesq approximation, part 1

$\rho = \overline{\rho}$ except when multiplied with the acceleration of gravity g

$$\overline{\rho}\frac{Du}{Dt} = \frac{\partial}{\partial t}\left(\overline{\rho}\left[\overline{u} + u'\right]\right) + \nabla \cdot \left(\overline{\rho}\left[\overline{u}\overline{\vec{V}} + u'\vec{V}' + \overline{u}\vec{V}' + u'\overline{\vec{V}}\right]\right) = -\left(\frac{\partial \overline{p}}{\partial x} + \frac{\partial p'}{\partial x}\right) + \overline{\rho} f\left(\overline{v} + v'\right) + \overline{F}_x + F_x' \quad (11)$$

Taking the average of the U-momentum equation, utilizing that the average of terms containing only one fluctuation parameter is zero and utilizing mass continuity for the mean flow, yields

$$\begin{aligned}
\overline{\frac{Du}{Dt}} &= \frac{\partial \overline{u}}{\partial t} + \overline{\vec{V}} \cdot \nabla \overline{u} + \overline{\rho}^{-1} \nabla \cdot \left(\overline{\rho} \overline{\vec{V}' u'} \right) \\
&= -\overline{\rho}^{-1} \frac{\partial \overline{p}}{\partial x} + f \overline{v} + \overline{\rho}^{-1} \overline{F}_x \qquad (12)
\end{aligned}$$

(12) is the mean U-momentum equation. Similarly the mean V and W equations become

$$\begin{aligned}
\overline{\frac{Dv}{Dt}} &= \frac{\partial \overline{v}}{\partial t} + \overline{\vec{V}} \cdot \nabla \overline{v} + \overline{\rho}^{-1} \nabla \cdot \left(\overline{\rho} \overline{\vec{V}' v'} \right) \\
&= -\overline{\rho}^{-1} \frac{\partial \overline{p}}{\partial y} - f \overline{u} + \overline{\rho}^{-1} \overline{F}_y \qquad (13)
\end{aligned}$$

$$\begin{aligned}
\overline{\frac{Dw}{Dt}} &= \underline{\frac{\partial \overline{w}}{\partial t} + \overline{\vec{V}} \cdot \nabla \overline{w}} + \overline{\rho}^{-1} \nabla \cdot \left(\overline{\rho} \overline{\vec{V}' w'} \right) \\
&= \underline{-\overline{\rho}^{-1} \frac{\partial \overline{p}}{\partial z} - g \frac{\rho}{\overline{\rho}}} + \overline{\rho}^{-1} \overline{F}_z \\
&= \overline{\rho}^{-1} \overline{F}_z \qquad (14)
\end{aligned}$$

These equations follow by taking the Reynold average of the instantaneous equations for V and W in (15) and (16) below.

$$\begin{aligned}\overline{\rho}\frac{Dv}{Dt} &= \frac{\partial}{\partial t}\left(\overline{\rho}\left[\overline{v}+v'\right]\right)+ \\ \nabla\cdot\left(\overline{\rho}\left[\overline{v}\overline{\vec{V}}+v'\vec{V}'+\overline{v}\vec{V}'+v'\overline{\vec{V}}\right]\right) &= \\ -\left(\frac{\partial\overline{p}}{\partial y}+\frac{\partial p'}{\partial y}\right)-\overline{\rho}f(\overline{u}+u')+\overline{F}_y+F_y^{'} & \qquad (15)\end{aligned}$$

$$\begin{aligned}\overline{\rho}\frac{Dw}{Dt} &= \frac{\partial}{\partial t}\left(\overline{\rho}\left[\overline{w}+w'\right]\right)+ \\ \nabla\cdot\left(\overline{\rho}\left[\overline{w}\overline{\vec{V}}+w'\vec{V}'+\overline{w}\vec{V}'+w'\overline{\vec{V}}\right]\right) &= \\ -\left(\frac{\partial\overline{p}}{\partial z}+\frac{\partial p'}{\partial z}\right)-\overline{\rho}g(1+\frac{\rho'}{\overline{\rho}})+\overline{F}_z+F_z^{'} & \qquad (16)\end{aligned}$$

HIRLAM: (14) is based on **hydrostatic balance for the mean flow**.
HIRLAM: Note that hydrostatic balance for the mean flow implies that the mean molecular friction $\overline{F}_z$ is neglected in (14).
HIRLAM: The mean molecular friction terms $\overline{F}_x$ in (12) and $\overline{F}_y$ in (13) are also neglected.

The **second part of the Boussinesq approx.: Desity fluctuations due to pressure fluctuations are negligible.**
can be used to rewrite (16) as

$$\overline{\rho}\frac{Dw}{Dt} = \frac{\partial}{\partial t}\left(\overline{\rho}\left[\overline{w} + w'\right]\right) +$$
$$\nabla \cdot \left(\overline{\rho}\left[\overline{w}\overline{\vec{V}} + w'\vec{V}' + \overline{w}\vec{V}' + w'\overline{\vec{V}}\right]\right) =$$
$$-\left(\frac{\partial \overline{p}}{\partial z} + \frac{\partial p'}{\partial z}\right) + \overline{\rho} g \frac{\theta_v'}{\overline{\theta_v}} + \overline{F}_z + F_z' \qquad (17)$$

since, from the equation of state

$$d\rho = \frac{1}{R_d}\left(\frac{1}{T_v}dp - R_d\frac{\rho}{T_v}dT_v\right) \approx -\frac{\rho}{T_v}dT_v. \qquad (18)$$

Interpreting $d\rho = \rho - \overline{\rho}$ and $dT = T - \overline{T_v}$ gives

$$-\frac{\rho'}{\overline{\rho}} \approx \frac{T_v'}{\overline{T_v}} = \frac{\theta_v'}{\overline{\theta_v}}. \qquad (19)$$

HIRLAM: Turbulence (within a grid-cell) is horizontally homogeneous

$$
\begin{aligned}
\frac{\overline{D}\overline{u}}{Dt} &= -\overline{\rho}^{-1}\frac{\partial \overline{p}}{\partial x} + f\overline{v} \\
&\quad + \overline{\rho}^{-1}\frac{\partial}{\partial z}\left(\overline{\rho}\nu\frac{\partial \overline{u}}{\partial z} - \overline{\rho}\overline{u'w'}\right) \\
\frac{\overline{D}\overline{v}}{Dt} &= -\overline{\rho}^{-1}\frac{\partial \overline{p}}{\partial y} - f\overline{u} \\
&\quad + \overline{\rho}^{-1}\frac{\partial}{\partial z}\left(\overline{\rho}\nu\frac{\partial \overline{v}}{\partial z} - \overline{\rho}\overline{v'w'}\right) \\
0 &\approx \overline{\rho}^{-1}\frac{\partial \overline{p}}{\partial z} - g \qquad (20)
\end{aligned}
$$

Note that mean molecular friction (approximated by neglecting the terms containing spatial gradients of the divergence) has been reintroduced in the horizontal momentum equations. This is done to show the analouge form of turbulent friction in HIRLAM.

HIRLAM: In the derivation of the turbulent kinetic energy equation the contribution to dissipation of TKE from spatial variations in $\nabla \cdot \vec{V}'$ is also neglegted.

HIRLAM:
Parameterization of turbulent momentum fluxes is an analogue to molecular diffusion

$$
\begin{aligned}
-\overline{u'w'} &= K_m \frac{\partial \overline{u}}{\partial z} \\
-\overline{v'w'} &= K_m \frac{\partial \overline{v}}{\partial z}
\end{aligned}
\tag{21}
$$

ν: **property of medium (gas/fluid)**
K_m: **property of flow; varies strongly in space and time**

Example: Monin-Obukhov surface layer similarity
$K_m = kzu_*/\phi_m(z/L)$

HIRLAM:

$$K_m = l_m\sqrt{\overline{e}} \tag{22}$$

$\overline{e} = 0.5\left(\overline{u'^2 + v'^2 + w'^2}\right)$ is the mean turbulent kinetic energy (TKE).

HIRLAM: l_m is **diagnostic** length scale of turbulence.

From (20) the local rate of change of mean momentum due to turbulence becomes

$$\begin{aligned} \frac{\partial\overline{u}}{\partial t} &= g^2\frac{\partial}{\partial p}\left[\overline{\rho}^2 K_m \frac{\partial\overline{u}}{\partial p}\right] \\ \frac{\partial\overline{v}}{\partial t} &= g^2\frac{\partial}{\partial p}\left[\overline{\rho}^2 K_m \frac{\partial\overline{v}}{\partial p}\right] \end{aligned} \tag{23}$$

Lower boundary condition

To increase numerical stability surface values

$$K_{mx} \approx -\overline{u'w'_s} z_n/\overline{u}$$
$$K_{my} \approx -\overline{v'w'_s} z_n/\overline{v}$$

are used instead of surface fluxes $-\overline{u'w'_s}$ and $-\overline{v'w'_s}$.

Knowledge of $\overline{e}$ is needed to solve (23).

Turbulent kinetic energy equation

Obtained by adding the equations in (24)

$$
\begin{aligned}
\overline{u'\left(\frac{Du}{Dt}-\overline{\frac{Du}{Dt}}\right)} &= \overline{u'\vec{V}'\cdot\nabla\overline{u} + u'\frac{\partial u'}{\partial t}} \\
&\quad + \overline{u'\overline{\vec{V}}\cdot\nabla u' + u'\vec{V}'\cdot\nabla u'} \\
&= \overline{-\overline{\rho}^{-1}u'\frac{\partial p'}{\partial x} + fu'v' + \nu u'\nabla^2 u'} \\
\overline{v'\left(\frac{Dv}{Dt}-\overline{\frac{Dv}{Dt}}\right)} &= \overline{v'\vec{V}'\cdot\nabla\overline{v} + v'\frac{\partial v'}{\partial t}} \\
&\quad + \overline{v'\overline{\vec{V}}\cdot\nabla v' + v'\vec{V}'\cdot\nabla v'} \\
&= \overline{-\overline{\rho}^{-1}v'\frac{\partial p'}{\partial y} - fu'v' + \nu v'\nabla^2 v'} \\
\overline{w'\left(\frac{Dw}{Dt}-\overline{\frac{Dw}{Dt}}\right)} &= \overline{w'\vec{V}'\cdot\nabla\overline{w} + w'\frac{\partial w'}{\partial t}} \\
&\quad + \overline{w'\overline{\vec{V}}\cdot\nabla w' + w'\vec{V}'\cdot\nabla w'} \\
&= \overline{-\overline{\rho}^{-1}w'\frac{\partial p'}{\partial z} + \theta_v' w'\frac{g}{\overline{\theta_v}} + \nu w'\nabla^2 w'}
\end{aligned}
\tag{24}
$$

By using incompressibility for the fluctuating part of the flow (implying $\nabla \cdot \vec{V}'=0$) the TKE equation becomes

$$\frac{\partial \overline{e}}{\partial t} + \overline{u}_j \frac{\partial \overline{e}}{\partial x_j} = \left[-\overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial j} \right]_S + \left[+ \frac{g}{\overline{\theta}_v} \overline{u_i' \theta_v'} \delta_{3i} \right]_B + \left[-\frac{\partial}{\partial x_j} (\overline{e u_j'}) \right]_T + \left[-\overline{\rho}^{-1} \frac{\partial}{\partial x_i} (\overline{p' u_i'}) \right]_T + \left[\nu \overline{u_i' \frac{\partial^2 u_i'}{\partial x_j'^2}} \right]_D \qquad (25)$$

$e = \frac{1}{2} \left(u'^2 + v'^2 + w'^2 \right)$

$\overline{e} = \frac{1}{2} \overline{\left(u'^2 + v'^2 + w'^2 \right)}$

Subscripts: **S** for shear production, **B** for buoyancy production, **T** for transport and **D** for dissipation.

HIRLAM: Since turbulence is assumed to be horizontally homogeneous within a grid volume, the local rate of change of mean TKE due to subgrid-scale turbulence is

$$\begin{aligned}\frac{\partial \overline{e}}{\partial t} = &\left[-\overline{u'w'}\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\frac{\partial \overline{v}}{\partial z}\right]_S + \\ &\left[+\frac{g}{\overline{\theta_v}}\overline{w'\theta_v'}\right]_B + \\ &-\left[\frac{\partial}{\partial z}\overline{ew'} + \overline{\rho}^{-1}\frac{\partial}{\partial z}\overline{p'w'}\right]_T + \\ &\left[\nu\left(\overline{u'\frac{\partial^2 u'}{\partial z^2}} + \overline{v'\frac{\partial^2 v'}{\partial z^2}} + \overline{w'\frac{\partial^2 w'}{\partial z^2}}\right)\right]_D \end{aligned} \tag{26}$$

$-\overline{w'w'}\partial\overline{w}/\partial z$ has been omitted, since it is much smaller than the retained terms in the shear production.

HIRLAM: The terms in the TKE equation are parameterized as

$$
\begin{aligned}
\left[-\overline{u'w'}\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\frac{\partial \overline{v}}{\partial z}\right]_S &\approx K_m \left[\left(\frac{\partial \overline{u}}{\partial z}\right)^2 + \left(\frac{\partial \overline{v}}{\partial z}\right)^2\right] \\
\left[+\frac{g}{\overline{\theta}_v}\overline{w'\theta'_v}\right]_B &\approx -\frac{g}{\overline{\theta}_v} K_h \frac{\partial \overline{\theta}_v}{\partial z} \\
-\left[\frac{\partial}{\partial z}\overline{ew'} + \overline{\rho}^{-1}\frac{\partial}{\partial z}\overline{p'w'}\right]_T &\approx \frac{\partial}{\partial z}\left(2K_m \frac{\partial \overline{e}}{\partial z}\right) \\
\left[\nu\left(\overline{u'\frac{\partial^2 u'}{\partial z^2}} + \overline{v'\frac{\partial^2 v'}{\partial z^2}} + \overline{w'\frac{\partial^2 w'}{\partial z^2}}\right)\right]_D &\approx -K_\epsilon \frac{\overline{e}}{l_\epsilon^2}
\end{aligned}
\tag{27}
$$

$K_m = l_m \sqrt{\overline{e}}$;
$K_h = l_h \sqrt{\overline{e}}$;
$K_\epsilon = l_\epsilon \sqrt{\overline{e}}$.
Note that there must be a negative sign in the parameterization of TKE-dissipation since γ' tends to be negatively correlated with $\partial^2\gamma'/\partial z^2$ (a local maximum and minimum in γ' tends to be positive and negative, respectively).

Specification of turbulence length scales l_m, l_h and l_ϵ is a challenge - ongoing work in HIRLAM.

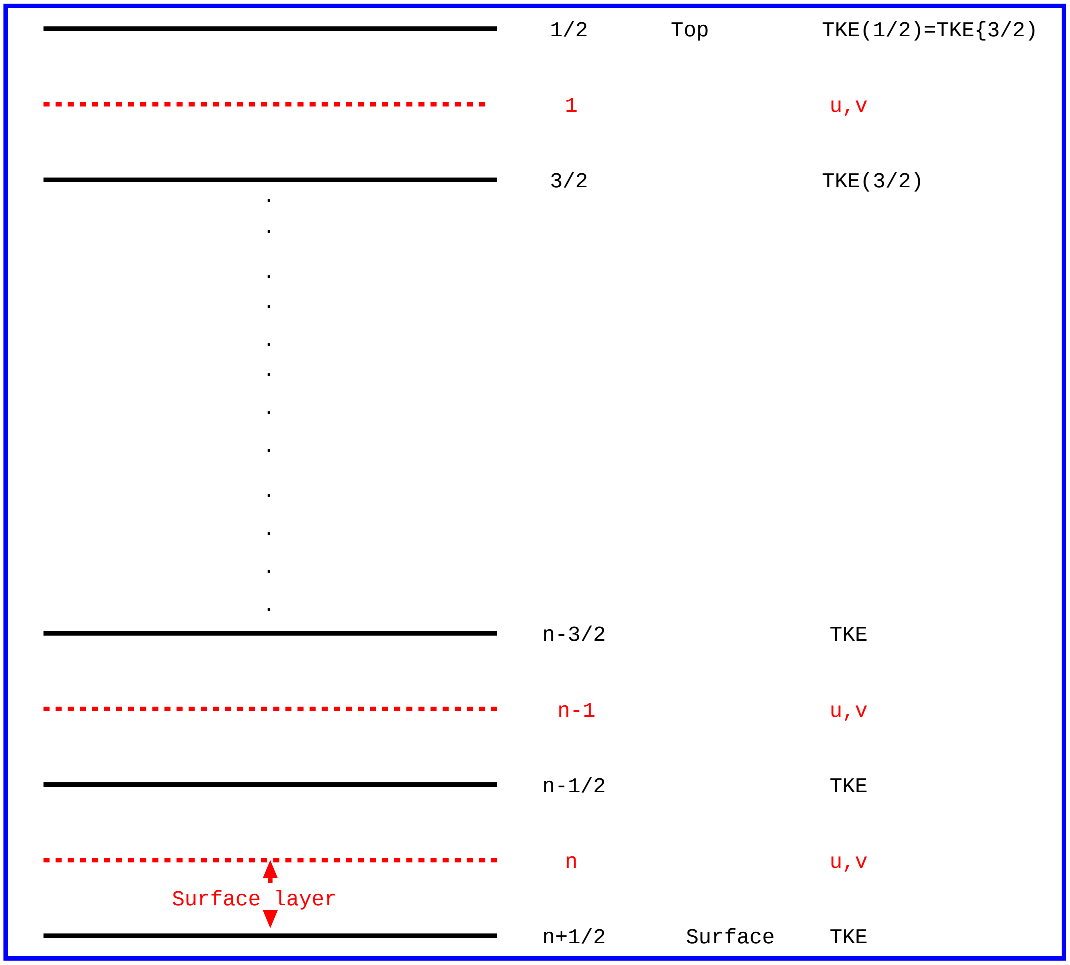

Figure 1: Schematic model levels in HIRLAM: Half levels (black), full levels (dotted red). Currently $n = 40$.

HIRLAM: Advection of $\overline{e}$ at half levels is done by winds at full levels!!!

Lower boundary condition for $\overline{e}$

HIRLAM: $\overline{e}$ is constant in the surface layer and calculated by

$$\overline{e}_{n+1/2} = \left(3.75 + \delta_u\left(-\frac{z_n}{L}\right)^{2/3}\right) {u_*}^2 + 0.2\delta_u {w_*}^2 \tag{28}$$

δ_u is 1 or 0 in ustable and stable stratification, respectively.

The TKE-equation (27) is partly implicit in time and solved for $\overline{e}$ at levels $i = n - \frac{1}{2}$ to $i = \frac{1}{2}$ with surface input from (28).

Calculation of $\overline{e}$ in the surface layer ((28)) requires knowledge of the turbulent kinematic surface fluxes of heat $(-\overline{w'\theta'_{vs}})$, moisture $(-\overline{w'q'}_s)$ and momentum $(\left(\overline{-u'w'_s}^2 + \overline{-v'w'_s}^2\right)^{1/2})$ since

$$L = -\frac{u_*^2}{\theta_{v*}}\frac{\overline{\theta_v}}{kg} \tag{29}$$

$$w_* = \left(\frac{g}{\overline{\theta_v}}\overline{w'\theta'_{vs}}\right)^{1/3} \tag{30}$$

Note: In (28) to (30)

$u_*^2 = \left(\overline{-u'w'_s}^2 + \overline{-v'w'_s}^2\right)^{1/2}$

$\overline{-w'\theta'_{vs}} = u_*\theta_{v*}$

Calculation of kinematic surface fluxes
$\gamma = u, v, \theta, q, ql, ...$
Exchange coefficient method

$$\begin{aligned}
\overline{w'\gamma'} &= C_\gamma \Delta\gamma |\vec{V}_n| \\
C_\gamma &= C_{\gamma N} \Psi_\gamma \left(Ri, \frac{z_n}{z_{0m}}, \frac{z_n}{z_{0\gamma}} \right) \\
C_{\gamma N} &= C_{mN} \left(1 + \ln \frac{z_{0m}}{z_{0\gamma}} / \frac{z_n}{z_{0m}} \right)^{-1} \\
C_{mN} &= k^2 \left(\ln \frac{z_n}{z_{0m}} \right)^{-2} \qquad (31)
\end{aligned}$$

In (31) $\Delta\gamma = \gamma_s - \gamma_n$ and Ψ_γ is a stability function.

The surface layer bulk Richardson number is used as a dynamic stability parameter in place of z_n/L.

Unstable straticication

$$\Psi_\gamma = 1 + \frac{a_{\gamma U} Ri}{1 + b_{\gamma U} C_{\gamma N} \left(-Ri \frac{z_n}{z_{0m}}\right)^{1/2}} \qquad (32)$$

$a_{mU} = 10, a_{hU} = a_{qU} = 15,$
$b_{mU} = b_{hU} = b_{qU} = 75$

Stable stratification

$$\begin{aligned} \Psi_m &= \frac{1}{1+\frac{a_{mS} Ri}{\sqrt{1+b_{mS} Ri}}} \\ \Psi_\gamma &= \frac{1}{1 + a_{\gamma S} Ri \sqrt{1 + b_{\gamma S} Ri}} \end{aligned} \qquad (33)$$

$a_{mS} = a_{hS} = a_{qS} = 10,$
$b_{mS} = b_{hS} = b_{qS} = 1$

Rotation of surface stress

The horizontal mean momentum equations for the steady state, horizontally homogeneous barotropic PBL are

$$
\begin{aligned}
0 &= f(v - v_{g0}) - \frac{\partial \overline{u'w'}}{\partial z} = f v_a + F_{tx} \\
0 &= -f(u - u_{g0}) - \frac{\partial \overline{v'w'}}{\partial z} = -f u_a + F_{ty}
\end{aligned}
\tag{34}
$$

Turbulent friction: $\vec{F}_t = (-\partial/\partial z(\overline{u'w'})\vec{i} - \partial/\partial z(\overline{v'w'})\vec{j})$.
Surface stress: $\vec{\tau} = (-\overline{u'w'}\vec{i} - \overline{v'w'}\vec{j})$.
Wind: $\vec{V} = u\vec{i} + v\vec{j}$.
Geostrophic wind: $\vec{V}_{g0} = u_{g0}\vec{i} + v_{g0}\vec{j}$.

Manipulation of (34) yields

$$\langle \vec{F}_t \rangle = f\vec{k} \times \langle \vec{V}_a \rangle = -h^{-1}\vec{\tau}_s \tag{35}$$

$$w(h) = f^{-1}\vec{k} \cdot \nabla \times \vec{\tau}_s \tag{36}$$

$$|\langle \vec{V}_a \rangle| \cos \alpha_F = \frac{1}{fh} |\vec{\tau}_s| \cos \alpha_F \tag{37}$$

$$F_A = fV_{g0} |\langle \vec{V}_a \rangle| \cos \alpha_F = V_{g0} |\langle \vec{F}_t \rangle| \cos \alpha_F \tag{38}$$

$\langle \vec{F}_t \rangle$: PBL mean turbulent frictional force per unit mass.
$\langle \vec{V}_a \rangle$: PBL mean ageostrophic wind.
α_F: Angle between surface stress and geostrophic wind.
$|\langle \vec{V}_a \rangle| \cos \alpha_F$: mean ageostrophic wind in the direction perpendicular to $\vec{V}_{g0}$ - proportional to the net cross isobaric mass flow in the PBL.
(38) states that the rate of work done by the frictional force is equal in magnitude to the rate of work done by the horizontal pressure gradient force.

Note: According to (37) the net cross isobaric mass flow in the PBL is proportional to the component of the surface stress along the geostrophic wind.

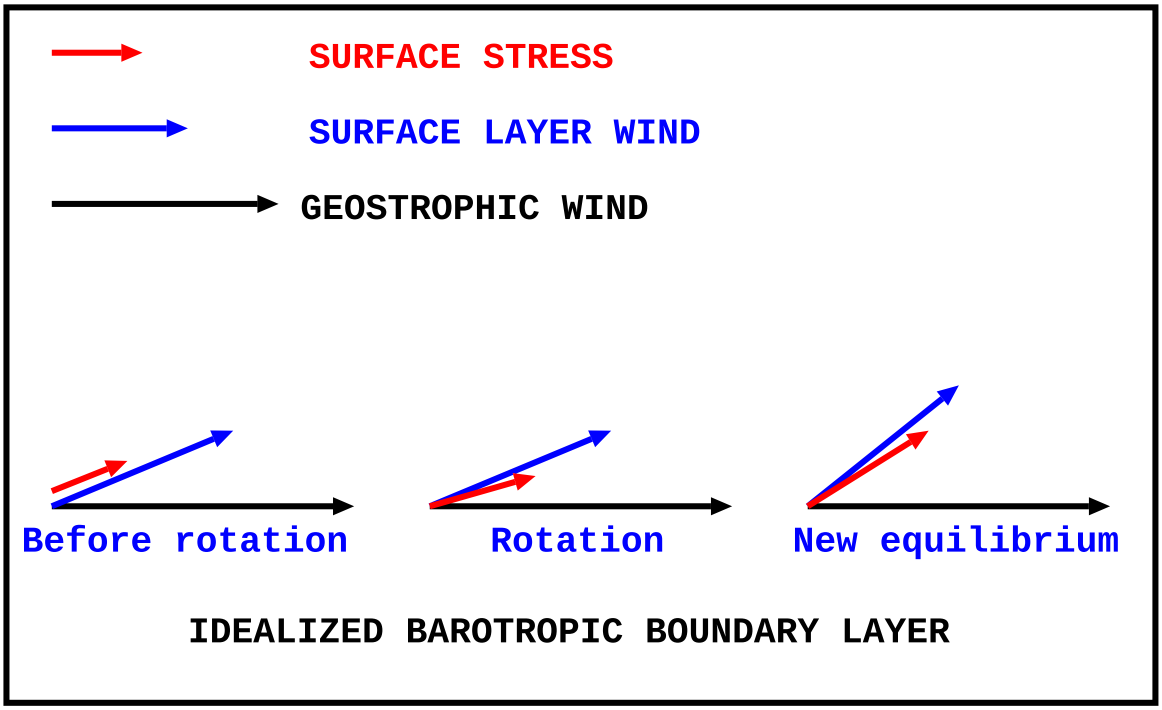

Figure 2: Schematic picture of the effect of a clockwise rotation of the surface stress in the Northern Hemisphere

It follows from (35) to (38) (Nielsen and Sass, 2005) that the response to clockwise rotation of the surface stress away from the surface layer wind is:

an **increase in surface stress**
an **increase in surface cross isobar angle**
$\Delta\alpha$: Angle the surface stress is rotated clockwise from surface layer wind

Currently $\cos\Delta\alpha = 1 - \left(\frac{Ri_*}{1+ari_*}\right)^\gamma$ with $Ri_* = Ri$, $\gamma = 1$ and $a = (1-0.9)^{-1/\gamma}$. If $Ri_* < 0$ then $\Delta\alpha = 0$

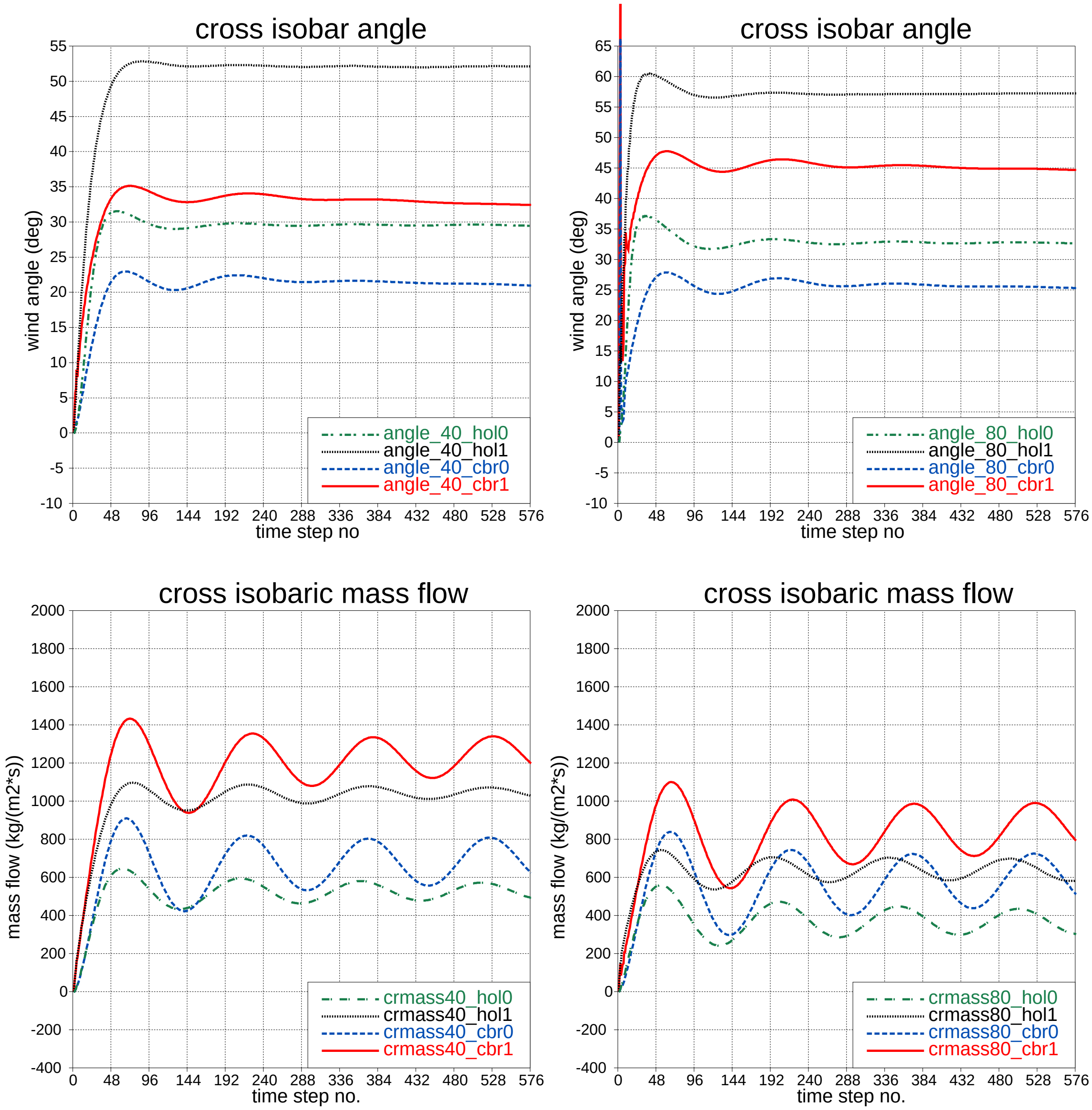

Figure 3: Variation with forecast lead time of surface cross isobar angle (top row) and cross isobaric mass flow (bottom row) in 1D-DMI-HIRLAM for barotropic conditions. Subscripts hol0 and cbr0 are for runs with the Holtslag scheme and the CBR scheme, respectively. Subscripts hol1 and cbr1 are for the same schemes with a clockwise rotation of the surface stress relative to the surface layer wind (see text). Subscripts 40 and 80 denote 40 and 80 vertical model levels, respectively. The location is at 70° N and the runs start from 00 UTC on 20 December with $z_0 = 0.01\,m$, $V_g = 10\,ms^{-1}$, $T_s = 10^\circ\,C$ (surface temperature), a lapse rate $0.009 Km^{-1}$ up to 1500 m and isothermal conditions above. The initial relative humidity is 20% and constant with height. Time step 576 corresponds to 48 hours. Note the 4 inertial cycles in the cross isobaric mass flow and the more rapid damping of the corresponding oscillation in α_0.

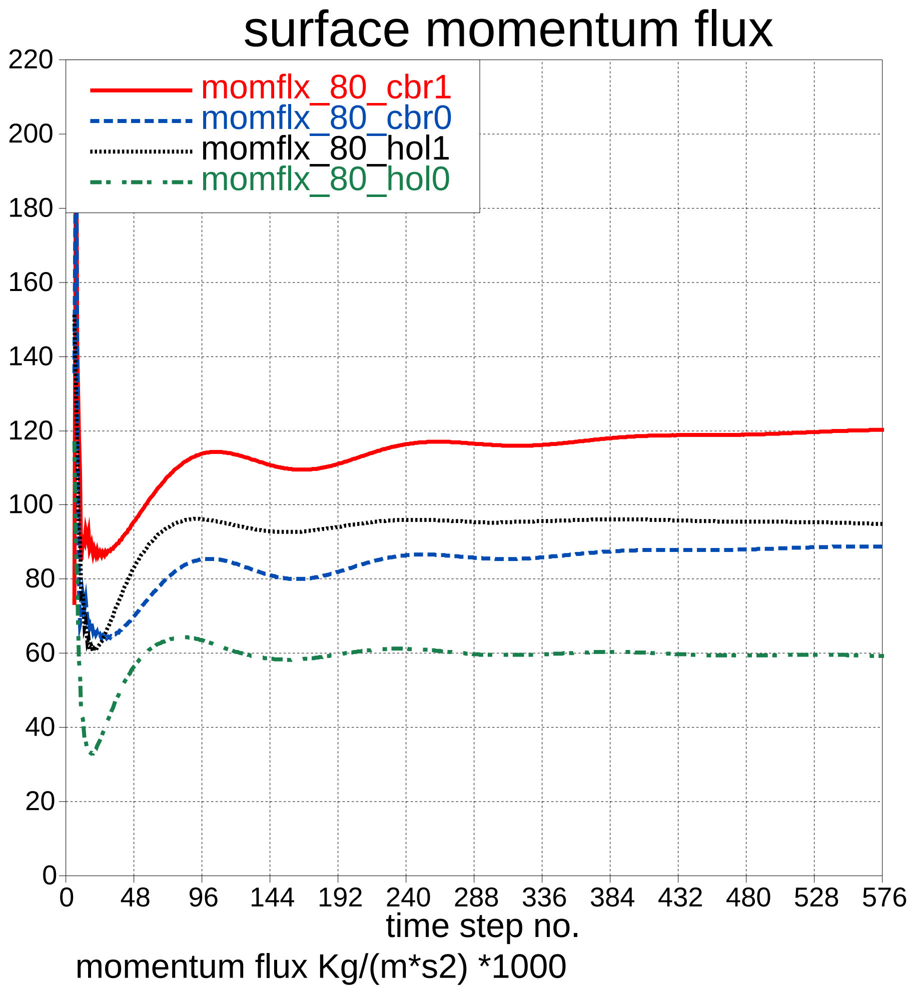

Figure 4: Evolution with time of the surface momentum flux. Initial conditions and meaning of subscripts are the same as in Figure 5.

Turbulent length scales

Stable stratification

$l_m = \left(l_{ms}^2 + l_{mb}^2\right)^{1/2}$

$l_h = \left(l_{hs}^2 + l_{hb}^2\right)^{1/2}$

$l_{ms} = e_m \frac{\sqrt{e}}{N}; e_m = 0.1 f(Ri, p)$

$l_{hs} = e_h \frac{\sqrt{e}}{N}; e_h = 0.1$

Unstable stratification

$l_m = l_{mb}; l_h = l_{hb}$

....

$l_{mb} = \left(\lambda^2 + l_{mi}^2\right)^{1/2}$

$l_{hb} = \left(\lambda^2 + l_{hi}^2\right)^{1/2}$

$\lambda^{-1} = \frac{1}{akz/2} + \frac{1}{\lambda_0}; \lambda_0 = 75\,m$

$l_{mi} = \left(\frac{1}{l_{mup}} + \frac{1}{l_{mdw}}\right)^{-1}$

$l_{hi} = \left(\frac{1}{l_{hup}} + \frac{1}{l_{hdw}}\right)^{-1}$

$l_{\gamma up} = \int_{z=0}^{z} F_\gamma(Ri) dz'; \gamma = m, h$

$l_{\gamma dw} = \int_{z}^{z_{top}} F_\gamma(Ri) dz'; \gamma = m, h$

$l_\epsilon = \frac{1}{3.75^2} l_m \approx 0.07 l_m$

For more details: See Unden et al. 2002

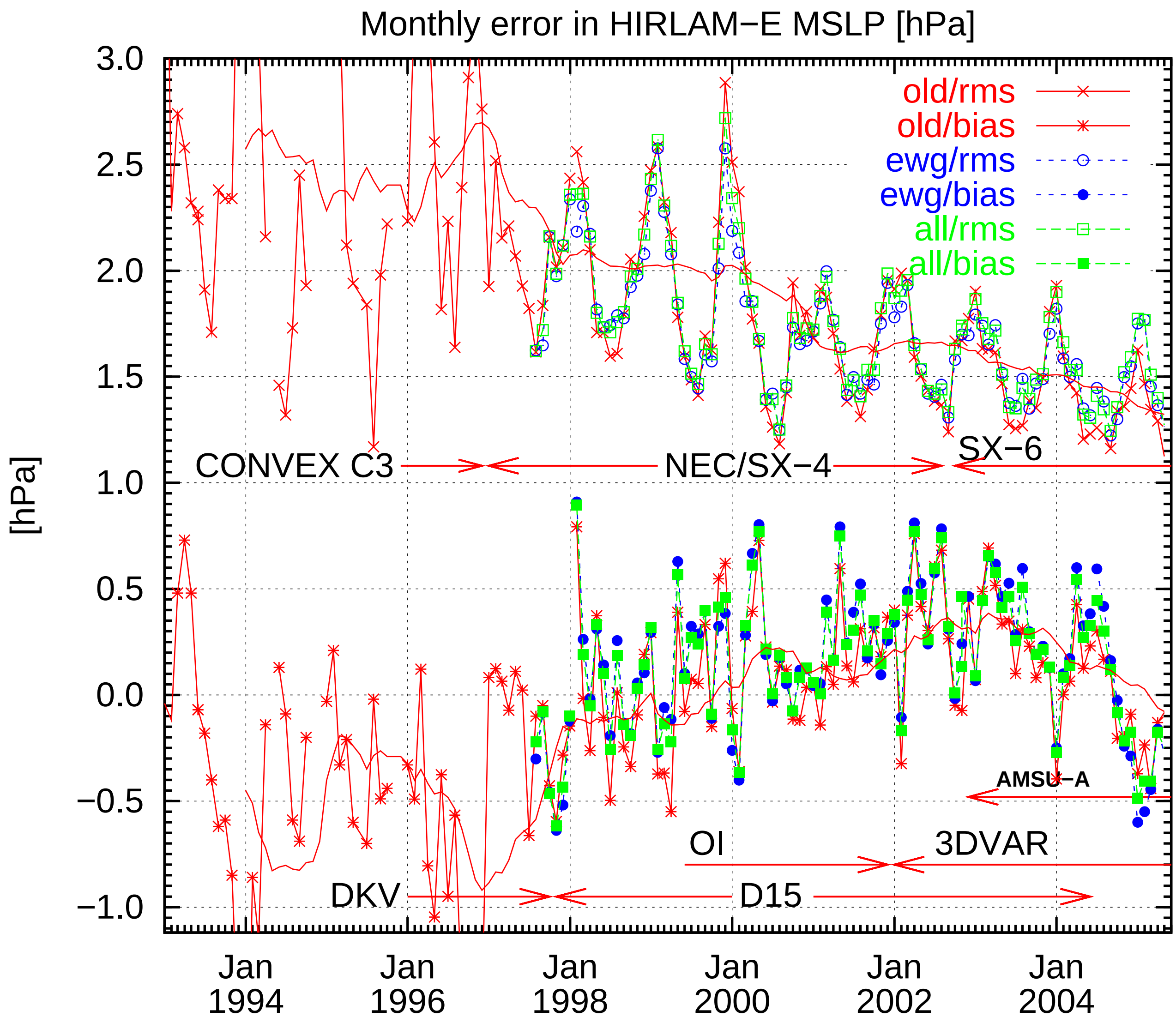

Figure 5: HIRLAM-NWP at DMI since 1992. Evolution with time of (upper part) root mean square error (rms) of mean sea level pressure (mslp) and (bottom part) bias of mslp evaluated against european observations (EWGLAM station list). The marked decrease in rms in 2000 is believed to be due mainly to the introduction of reanalyses with longer observation cutoff and blending of large scales from ECMWF into the HIRLAM analyses. The decrease in rms in recent years is a combined effect of improvements in in use of observations, analysis method and model dynamics and model physics, the latter including surface stress rotation since mid-2004. Forecast lead time is 24 hours.

References

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