Boundary Layer Aspects in Shallow Water Approximations

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SHALLOW-WATER FLOWS

Shallow-water flow is one of many special forms in which hydrodynamics presents itself

The general characteristics of such flows is that the vertical dimension is much smaller than any typical horizontal scale

Shallow –water flows are nearly horizontal, which allows considerable simplification in the mathematical formulations by assuming the pressure distribution to be hydrostatic

APPLICATIONS

Atmospheric flows, first attempt to predict weather

Tidal flows in ocean

Flows around structures, dambreak waves, coastal flows

Tsunamis generated by undersea landslides or earthquakes

Shallow water equations describing incompressible heavy fluid flow with a free surface on flat plate coincide with those of polytropic gas with the specific heat ratio equals to two

This allows transforming to the classical shallow water theory all continuous solutions for ideal gas equations

This analogy, however, do not apply to discontinuous shallow water flows because of the difference in the conditions on discontinuity surface for this problem

Shallow water flows on plane surface in the presence of the weak vertical inhomogeneities in the initial conditions will be considered

Shallow water approximation for the complete set of hydrodynamic equations in this case contains additional terms appearing as the result of depth averaging of the nonlinear terms in the initial fluid equations and normally has been relaxed in traditional derivation

This new terms describe an advective transport of impulse as a result of the dependence of horizontal shallow water flows on vertical coordinate

Two-dimensional incompressible Euler equations for a free surface

$$
\begin{cases}
\dfrac{\partial \rho}{\partial \mathrm{t}} + \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} = 0 \\
\dfrac{\partial V_x}{\partial t} + V_x \dfrac{\partial V_x}{\partial x} + V_y \dfrac{\partial V_x}{\partial y} + \dfrac{1}{\rho}\dfrac{\partial P}{\partial x} = 0 \\
\dfrac{\partial V_y}{\partial t} + V_x \dfrac{\partial V_y}{\partial x} + V_y \dfrac{\partial V_y}{\partial y} + \dfrac{1}{\rho}\dfrac{\partial P}{\partial y} + g = 0
\end{cases}
$$

SURFACE AND BOTTOM BOUNDARY CONDITIONS

The kinematic conditions say that water particles will not cross either boundary

For solid bottom this means that normal velocity component must wanish

For free surface relative normal velocity must vanish

Boundary conditions and hydrostatic pressure

$$\begin{cases} V_y\big|_{y=0} = 0 \\ V_y\big|_{y=h} = \dfrac{dh}{dt} \end{cases}$$

$$P(y) = P_A + \rho g(h - y)$$

DEPTH INTEGRATION

$$\begin{cases} \dfrac{\partial}{\partial t}\displaystyle\int_0^h \rho dy + \dfrac{\partial}{\partial x}\displaystyle\int_0^h V_x dy = 0 \\ \dfrac{\partial}{\partial t}\displaystyle\int_0^h V_x dy + \dfrac{\partial}{\partial x}\displaystyle\int_0^h V_x^2 dy + gh\dfrac{\partial h}{\partial x} = 0 \end{cases}$$

,

$$V_x = u + V_x'(x, y, t)$$

$$\int_0^h V_x'(x,y,t)dy = 0$$

$$u = \frac{1}{h}\int_0^h V_x(x, y, t)dy$$

MODIFIED SHALLOW-WATER EQUATIONS

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0 \\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}(hu^2) + gh\dfrac{\partial h}{\partial x} + \dfrac{\partial}{\partial x}\displaystyle\int_0^h \left(V_x'\right)^2 dy = 0 \end{cases}$$

Extended S-W Equations

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0 \\ \dfrac{\partial}{\partial t}(hu) + \dfrac{\partial}{\partial x}(hu^2) + gh\dfrac{\partial h}{\partial x} + \dfrac{\partial}{\partial x}R = 0 \end{cases}$$

$$R = \int_0^h \left(V_x'\right)^2 dy$$

MODYFIED SHALLOW-WATER EQUATIONS

Simple analytical model

$$R=\int_0^h \left(V_x'\right)^2 dy=\left(V_x'\Big|_{z=z_0(x,t)}\right)^2 h$$

$$\frac{\partial}{\partial x}R=\frac{\partial}{\partial x}\int_z^h \left(V_x'\right)^2 dy \approx k_0\frac{\partial h}{\partial x}=gH\frac{\partial h}{\partial x}$$

$$z_0(x,t)\approx const=z_0 \Rightarrow \left(V_x'\Big|_{z=z_0}\right)^2 \approx const=k_0$$

$$\begin{cases} \dfrac{\partial h}{\partial t}+u\dfrac{\partial h}{\partial x}+h\dfrac{\partial u}{\partial x}=0 \qquad , \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+g\dfrac{(h+H)}{h}\dfrac{\partial h}{\partial x}=0. \end{cases}$$

EQUATIONS IN RIEMANN INVARIANTS

Final form of equations

;

$$\frac{\partial s}{\partial t}+(u-c)\frac{\partial s}{\partial x}=0 \qquad s=u-2c-\tilde{c}\ln\left|\frac{c-\tilde{c}}{c+\tilde{c}}\right|$$

,

$$\frac{\partial r}{\partial t}+(u+c)\frac{\partial r}{\partial x}=0 \qquad r=u+2c+\tilde{c}\ln\left|\frac{c-\tilde{c}}{c+\tilde{c}}\right|$$

,

.

$$c=\sqrt{g(H+h)} \qquad \tilde{c}=\sqrt{gH}$$

BASIC CONTINUOUS SOLUTIONS

Traveling Riemann Waves

$$\begin{cases} u(x,t) - 2c(x,t) - \tilde{c}\ln\left(\dfrac{c(x,t)-\tilde{c}}{c(x,t)+\tilde{c}}\right) = u(x,0) - 2c(x,0) - \tilde{c}\ln\left(\dfrac{c(x,0)-\tilde{c}}{c(x,0)+\tilde{c}}\right), \\ \dfrac{dx}{dt} = u - c. \end{cases}$$

$$\begin{cases} u(x,t) + 2c(x,t) + \tilde{c}\ln\left(\dfrac{c(x,t)-\tilde{c}}{c(x,t)+\tilde{c}}\right) = u(x,0) + 2c(x,0) + \tilde{c}\ln\left(\dfrac{c(x,0)-\tilde{c}}{c(x,0)+\tilde{c}}\right), \\ \dfrac{dx}{dt} = u + c. \end{cases}$$

BASIC CONTINUOUS SOLUTIONS

Centered Riemann Waves

$$\begin{cases} u(x,t) - 2c(x,t) - \tilde{c}\ln\left(\dfrac{c(x,t)-\tilde{c}}{c(x,t)+\tilde{c}}\right) = u(x_0,t_0) - 2c(x_0,t_0) - \tilde{c}\ln\left(\dfrac{c(x_0,t_0)-\tilde{c}}{c(x_0,t_0)+\tilde{c}}\right) \\ \dfrac{x-x_0}{t-t_0} = u - c. \end{cases},$$

$$\begin{cases} u(x,t) + 2c(x,t) + \tilde{c}\ln\left(\dfrac{c(x,t)-\tilde{c}}{c(x,t)+\tilde{c}}\right) = u(x_0,t_0) + 2c(x_0,t_0) + \tilde{c}\ln\left(\dfrac{c(x_0,t_0)-\tilde{c}}{c(x_0,t_0)+\tilde{c}}\right) \\ \dfrac{x-x_0}{t-t_0} = u + c. \end{cases},$$

INTEGRAL FORM OF EQUATIONS

We rewrite modified SW equations

$$\begin{cases} \iint\limits_G \left(\dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} \right) dG = 0, \\ \iint\limits_G \left(\dfrac{\partial hu}{\partial t} + \dfrac{\partial hu^2}{\partial x} + {}^{g}\!/_{2} \dfrac{\partial h^2}{\partial x} + gH \dfrac{\partial h}{\partial x} \right) dG = 0. \end{cases}$$

HUGONOIOT CONDITIONS IN TRANSITION DOMAIN.

The discontinuity conditions are:

$$[u] = u_2(t) - u_1(t),$$
$$[h] = h_2(t) - h_1(t),$$

$$D[h] = [hu],$$
$$D[hu] = \left[h\, u^2 + \frac{g}{2} h^2 + gHh\right].$$

DISCONTINUOUS SOLUTIONS

The relationship between the values of the main functions at the discontinuity line and the velocity of a discontinuity itself :

$$D = \frac{h_1 u_1 - h_2 u_2}{h_1 - h_2},$$

$$\frac{\left(h_1 u_1 - h_2 u_2\right)^2 - (h_1 u_1^2 - h_2 u_2^2)\left(h_1 - h_2\right)}{(h_1 - h_2)} = gH(h_1 - h_2) + \frac{g}{2}(h_1^2 - h_2^2),$$

$$u_1 = u_2 - (h_1 - h_2)\sqrt{\frac{g}{2}\frac{(h_1 + h_2 + 2H)}{h_1 h_2}}$$

COMPARATIVE ANALYSIS OF OBTAINED SOLUTIONS

$$c(H,x,t) \approx \sqrt{g(h+\varepsilon)}\left(1+\frac{(H-\varepsilon)}{2h}-\frac{(H-\varepsilon)^2}{8h^2}+\frac{(H-\varepsilon)^3}{16h^3}\right)$$

$$[u](H,x(t)) \approx (h_2-h_1)\sqrt{g/2\frac{(h_2+h_1+2\varepsilon)}{h_2h_1}}\left(1+\frac{H}{(h_2+h_1+2\varepsilon)}-1/2\frac{H^2}{(h_2+h_1+2\varepsilon)^2}+1/2\frac{H^3}{(h_2+h_1+2\varepsilon)^3}\right)$$

COMPARATIVE ANALYSIS OF OBTAINED SOLUTIONS

It is easy to see that for the continuous solutions the relation

$$I = \frac{H}{2h}$$

is the indicator of the limit of applicability of the classical shallow water equations. Similarly, for the discontinuous solutions the dimensionless combination is the same as the indicator $I = \frac{H}{h_1 + h_2}$

or in initial designations: $I = \frac{H}{g}$

For the hydrodynamic flows in which $I << 1$

the use of the classical shallow water model is correct and full justified.

Riemann problem definition for modified shallow water equations

We rewrite equations.

As an initial Cauchy conditions we use an arbitrary piecewise constant initial conditions at t=0 for the left (x<0) and right (x>0) half-spaces

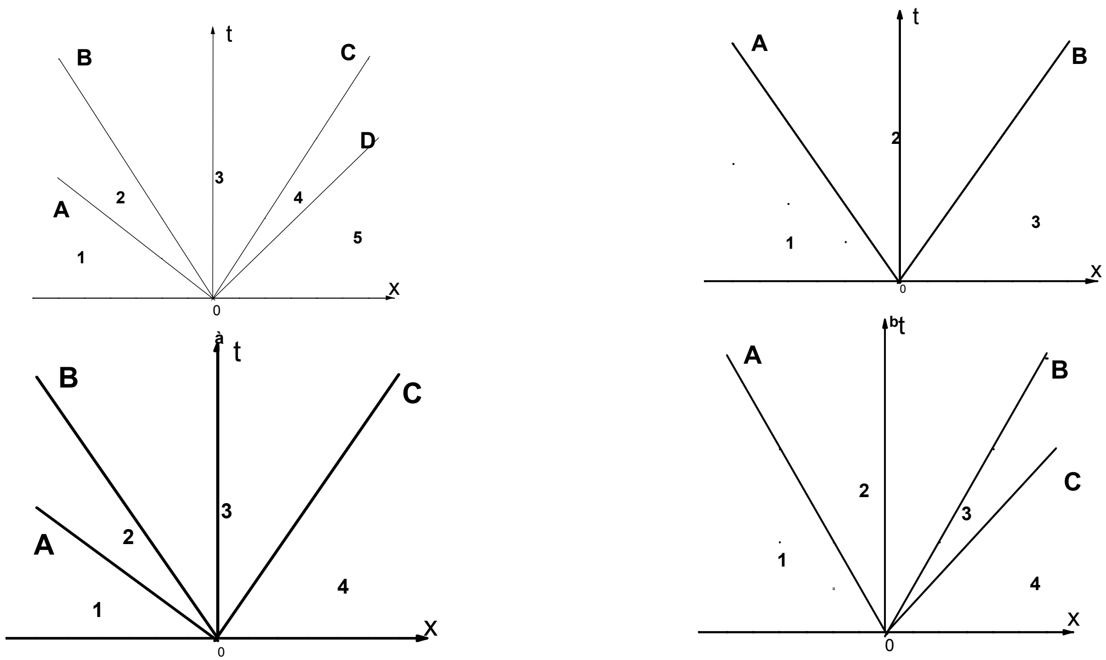
$$\begin{cases} \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} + h\dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + g\dfrac{(h+H)}{h}\dfrac{\partial h}{\partial x} = 0, \end{cases}$$

$$\begin{cases} u = u_1, h = h_1, x < 0, \\ u = u_2, h = h_2, x > 0. \end{cases}$$

POSSIBLE CONFIGURATIONS

A self-similar picture of originating flow on the plane x,t is schematically shown by the four possible configurations:

- “ two dilatation waves” (a)
- “ two hydrodynamics jumps” (b)
- “ dilatation wave - hydrodynamic jump ” (c)
- “ hydrodynamic jump - dilatation wave” (d)



c
d

TWO DILLATATION WAVES

$$u_1 < \varphi(c_2) - \varphi(c_1)$$

$$\varphi(\mathrm{c}) = 2c + \widetilde{\mathrm{c}}\,\ln\left(\frac{c - \widetilde{\mathrm{c}}}{c + \widetilde{\mathrm{c}}}\right)$$

TWO HYDRODYNAMIC JUMPS

$$u_1 > (h_1 - h_2)\sqrt{\frac{g}{2h_1h_2}(h_1 + h_2 + 2H)}$$

HYDRODYNAMIC JUMPS-DILLATATION WAVES

$$\varphi(c_2) - \varphi(c_1) < u_1 < (h_1 - h_2)\sqrt{\frac{g}{2h_1h_2}(h_1 + h_2 + 2H)}$$

PARTICULAR CONFIGURATIONS

In the case when $h_1 < h_2$, all changes occur only in the last configuration which is mirror image into a configuration "hydrodynamics jump – rarefaction wave" All the results are obtained by interchanging h_1 and h_2

POSSIBLE EXTENTIONS

$$\frac{1}{\tilde{h}}\frac{\partial}{\partial x}\int_{z}^{h} V_x'^2 dy =$$

$$= \frac{\partial}{\partial x}\left(k_0 \ln \tilde{h} + \sum_{i=1}^{+\infty} \frac{1}{i}\left(\left(i+1 \right) k_i \tilde{h}^i + \left(i-1 \right) k_{-i} \tilde{h}^{-i} \right) \right)$$

POSSIBLE EXTENTIONS

$$\int_0^h \left(V_x'\right)^2 dy = F(x,t,u,h)$$

$$c = \pm\sqrt{gh + F_h' + \left(\frac{F_u'}{2h}\right)^2} - \frac{F_u'}{2h}$$

$$gh + F_h' + \left(\frac{F_u'}{2h}\right)^2 > 0$$

$$-\frac{c}{h}\left(\frac{\partial h}{\partial t} + (u-c)\frac{\partial h}{\partial x}\right) + \frac{\partial u}{\partial t} + (u-c)\frac{\partial u}{\partial x} = -\frac{1}{h}\frac{\partial F}{\partial x}$$

$$\frac{c}{h}\left(\frac{\partial h}{\partial t} + (u+c)\frac{\partial h}{\partial x}\right) + \frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} = -\frac{1}{h}\frac{\partial F}{\partial x}$$

POSSIBLE EXTENTIONS

$$c/h = A(h)B(u)$$

$$\frac{\partial(\varphi_1 + \varphi_2)}{\partial t} + (u + c)\frac{\partial(\varphi_1 + \varphi_2)}{\partial x} = 0$$

$$\varphi_1(h) = \int A(h)dh$$

$$\frac{\partial(\varphi_2 - \varphi_1)}{\partial t} + (u - c)\frac{\partial(\varphi_2 - \varphi_1)}{\partial x} = 0$$

$$\varphi_2(u) = \int \frac{1}{B(u)} du$$

$$r = \varphi_1(h) + \varphi_2(u)$$

$$s = \varphi_2(u) - \varphi_1(h)$$

Gas-dynamic analogy for shallow water equations is generalized in the case when initial conditions depend on vertical coordinate

Simple parameterization of advective term allowing full theoretical analysis of the solutions of simple waves and Riemann problem for modified shallow water equations is suggested

The analytical simple wave solutions obtained have permitted to find dimensionless parameter that restricts limits of applicability of the classical shallow water equations and neglecting of the advective impulse transfer

Solution of the initial discontinuity decay problem for modified shallow water equations is found.

NONHOMOGENEOUS FLUID

$$\begin{cases}
\dfrac{\partial \rho V_x}{\partial x} + \dfrac{\partial \rho V_y}{\partial y} + \dfrac{\partial \rho V_z}{\partial z} = 0 \\
\rho \dfrac{\partial V_x}{\partial t} + \rho V_x \dfrac{\partial V_x}{\partial x} + \rho V_y \dfrac{\partial V_x}{\partial y} + \rho V_z \dfrac{\partial V_x}{\partial z} + \dfrac{\partial P}{\partial x} = 0 \\
\rho \dfrac{\partial V_y}{\partial t} + \rho V_x \dfrac{\partial V_y}{\partial x} + \rho V_y \dfrac{\partial V_y}{\partial y} + \rho V_z \dfrac{\partial V_y}{\partial z} + \dfrac{\partial P}{\partial y} = 0 \\
\dfrac{\partial V_z}{\partial t} + V_x \dfrac{\partial V_z}{\partial x} + V_y \dfrac{\partial V_z}{\partial y} + V_z \dfrac{\partial V_z}{\partial z} + \dfrac{1}{\rho} \dfrac{\partial P}{\partial z} + g = 0
\end{cases}$$

BOUNDARY CONDITIONS

Conditions on solid boundary and free surface

$$V_z\Big|_{z=0} = 0$$

$$V_z\Big|_{z=h} = \frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}V_x\Big|_{z=h} + \frac{\partial h}{\partial y}V_y\Big|_{z=h}$$

HYDROSTATIC PRESSURE

$$\frac{\partial P}{\partial z} + \rho(z) g = 0$$

,

$$P = -g \int_0^z \rho(z) dz$$

$$P\Big|_{z=h} = P_a = const$$

$$P = -g \int \rho(z) dz + \left(g \int \rho(z) dz \right)\Bigg|_{z=h} + P_a$$

Depth averaged equations

$$\frac{\partial}{\partial t}\int_0^h \rho V_x dz + \frac{\partial}{\partial x}\int_0^h \rho V_x^2 dz + \frac{\partial}{\partial y}\int_0^h \rho V_y V_x dz + \int_0^h \frac{\partial P}{\partial x} dz = 0$$

$$\frac{\partial}{\partial t}\int_0^h \rho V_y dz + \frac{\partial}{\partial x}\int_0^h \rho V_y^2 dz + \frac{\partial}{\partial y}\int_0^h \rho V_y V_x dz + \int_0^h \frac{\partial P}{\partial y} dz = 0$$

MASS AVERAGING

Velocity averaging

$$\left\langle V_x \right\rangle = \left\langle V_x \right\rangle(x,y,t) = \frac{1}{\int\limits_0^h \rho dz} \int\limits_0^h \rho V_x dz$$

$$\left\langle V_x \right\rangle = \left\langle V_x \right\rangle(x,y,t) = \frac{1}{\int\limits_0^h \rho dz} \int\limits_0^h \rho V_x dz$$

$$V_x = \left\langle V_x \right\rangle + V_x'(x,y,z,t) \qquad V_y = \left\langle V_y \right\rangle + V_y'(x,y,z,t)$$

SHALOW-WATER APPROXIMATION

$$\frac{\partial h}{\partial t}+\overline{V}_x\frac{\partial h}{\partial x}+\overline{V}_y\frac{\partial h}{\partial y}+\frac{\int_0^h \rho dy}{\rho(h)}\frac{\partial \overline{V}_x}{\partial x}+\frac{\int_0^h \rho dy}{\rho(h)}\frac{\partial \overline{V}_y}{\partial y}=0$$

$$\frac{\partial \overline{V}_x}{\partial t}+\overline{V}_x\frac{\partial \overline{V}_x}{\partial x}+\overline{V}_y\frac{\partial \overline{V}_x}{\partial y}+\frac{1}{\int_0^h \rho dz}\frac{\partial}{\partial x}\int_0^h \rho V_x'^2 dz+$$

$$+\frac{1}{\int_0^h \rho dz}\frac{\partial}{\partial y}\int_0^h \rho V_y' V_x' \, dz+\frac{1}{\int_0^h \rho dz}\int_0^h \frac{\partial P}{\partial h}dz\frac{\partial h}{\partial x}=0$$

SHALLOW-WATER APPROXIMATION

$$\frac{\partial \overline{V_y}}{\partial t} + \overline{V_x}\frac{\partial \overline{V_y}}{\partial x} + \overline{V_y}\frac{\partial \overline{V_y}}{\partial y} + \frac{1}{\int_0^h \rho dz}\frac{\partial}{\partial y}\int_0^h \rho {V'_y}^2 dz$$

$$+ \frac{1}{\int_0^h \rho dz}\frac{\partial}{\partial x}\int_0^h \rho V'_y V'_x \, dz + \frac{1}{\int_0^h \rho dz}\int_0^h \frac{\partial P}{\partial h} dz \frac{\partial h}{\partial y} = 0$$

1-D CASE

$$\frac{\partial h}{\partial t}+\overline{V}\,\frac{\partial h}{\partial x}+\frac{\int_0^h \rho dz}{\rho(h)}\,\frac{\partial}{\partial x}\overline{V}\;=0$$

$$\frac{\partial \overline{V}}{\partial t}+\frac{1}{\int_0^h \rho dz}\int_0^h \frac{\partial P}{\partial h}\,dz\,\frac{\partial h}{\partial x}+\overline{V}\,\frac{\partial \overline{V}}{\partial x}=0$$

EQUATIONS IN RIEMANN FORM

$$\frac{\partial}{\partial t}\left(\overline{V}+\int\frac{\rho(h)c}{\Omega(h)}dh\right)+\left(\overline{V}+c\right)\frac{\partial}{\partial x}\left(\overline{V}+\int\frac{\rho(h)c}{\Omega(h)}dh\right)=0$$

$$\frac{\partial}{\partial t}\left(\overline{V}-\int\frac{\rho(h)c}{\Omega(h)}dh\right)+\left(\overline{V}-c\right)\frac{\partial}{\partial x}\left(\overline{V}-\int\frac{\rho(h)c}{\Omega(h)}dh\right)=0$$

MODIFIED EQUATIONS

$$\frac{\partial H}{\partial t} + \overline{V}\,\frac{\partial H}{\partial x} + H\,\frac{\partial}{\partial x}\overline{V} = 0$$

$$\frac{\partial H}{\partial t} + \overline{V}\,\frac{\partial H}{\partial x} + H\,\frac{\partial}{\partial x}\overline{V} = 0$$

$$\frac{dH}{dh}\,\frac{\int_0^h \rho dz}{\rho(h)} = h$$

MODIFIED EQUATIONS

$$\frac{\partial h}{\partial t} + \overline{V}\,\frac{\partial h}{\partial x} + h\,\frac{\partial}{\partial x}\overline{V} = 0$$

$$\frac{\partial \overline{V}}{\partial t} + g\,\frac{\partial h}{\partial x} + \overline{V}\,\frac{\partial \overline{V}}{\partial x} + \frac{1}{h}\,\frac{\partial F}{\partial x} = 0$$

$$F = F(h) \quad \Rightarrow \quad \frac{\partial F}{\partial x} = \frac{dF}{dh}\,\frac{\partial h}{\partial x}$$

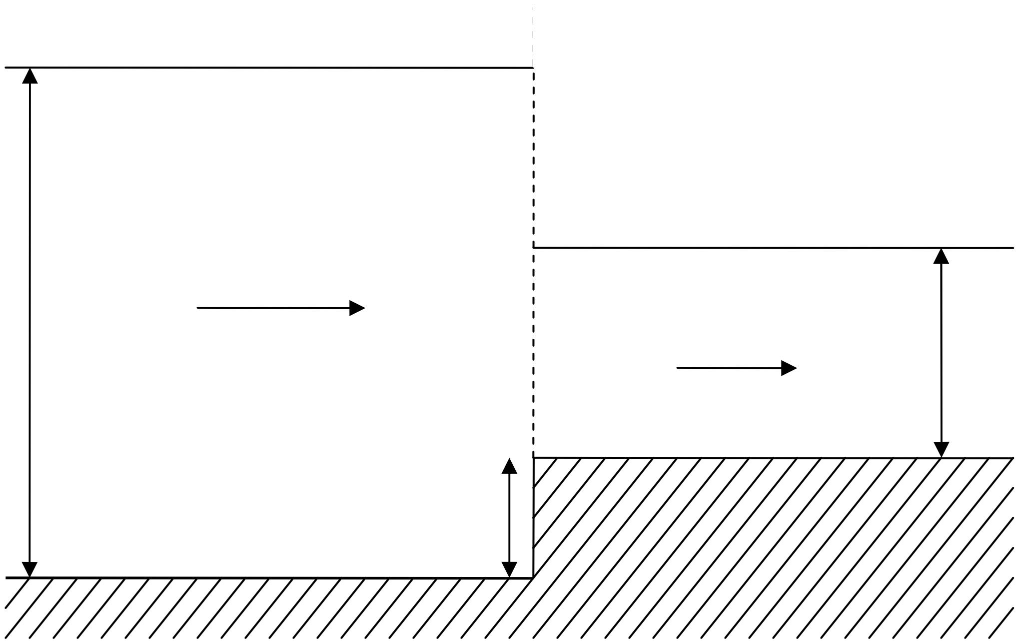
EXTENDED EQUATIONS

$$\int_0^h \rho V_x'^2 \, dy = F(x, t, \overline{V}, h)$$

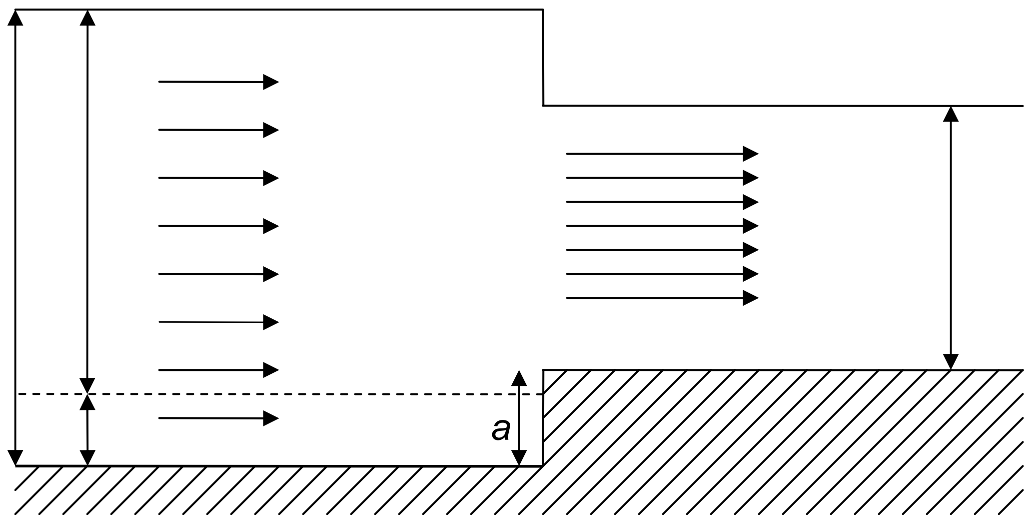
$$\frac{\partial h}{\partial t} + \overline{V}\,\frac{\partial h}{\partial x} + \frac{\Omega(h)}{\rho(h)}\,\frac{\partial}{\partial x}\overline{V} = 0$$

$$\frac{\partial \overline{V}}{\partial t} + gh\,\frac{\rho(h)}{\Omega(h)}\,\frac{\partial h}{\partial x} + \overline{V}\,\frac{\partial \overline{V}}{\partial x} + \frac{1}{\Omega(h)}\left(F_h'\,\frac{\partial h}{\partial x} + F_u'\,\frac{\partial \overline{V}}{\partial x} + \frac{\partial F}{\partial x} \right) = 0$$

SHALLOW-WATER FLOWS ON STEP



MODEL



2 LAYER MODEL

$$
\begin{cases}
\dfrac{\partial h_1}{\partial t} + \dfrac{\partial}{\partial x}(h_1 u_1) = 0 \\
\dfrac{\partial h_2}{\partial t} + \dfrac{\partial}{\partial x}(h_2 u_2) = 0 \\
\dfrac{\partial}{\partial t}(h_1 u_1) + \dfrac{\partial}{\partial x}(h_1 u_1^2 + \dfrac{1}{2} g h_1^2) + g h_1 \dfrac{\partial h_2}{\partial x} = 0 \\
\dfrac{\partial}{\partial t}(h_2 u_2) + \dfrac{\partial}{\partial x}(h_2 u_2^2 + \dfrac{1}{2} g h_2^2) + g h_2 \dfrac{\partial h_1}{\partial x} = 0
\end{cases}
$$

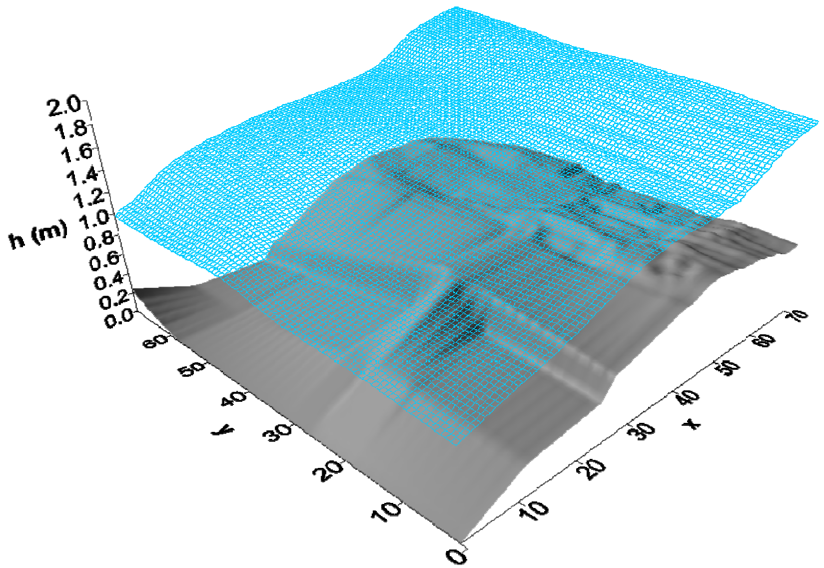
$$\begin{cases} \dfrac{\partial h_1}{\partial t} + \dfrac{\partial}{\partial x}(h_1 u_1) = 0 \\ \dfrac{\partial}{\partial t}(h_1 u_1) + \dfrac{\partial}{\partial x}(h_1 u_1^2 + \dfrac{1}{2} g h_1^2) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial h_2}{\partial t} + \dfrac{\partial}{\partial x}(h_2 u_2) = 0 \\ \dfrac{\partial}{\partial t}(h_2 u_2) + \dfrac{\partial}{\partial x}(h_2 u_2^2 + \dfrac{1}{2} g h_2^2) = 0 \end{cases}$$

$$h_2 = h_1 - h^*$$

$$
\begin{cases}
\dfrac{\partial h_1}{\partial t} + \dfrac{\partial}{\partial x}(h_1 u_1) = 0 \\[2ex]
\dfrac{\partial}{\partial t}(h_1 u_1) + \dfrac{\partial}{\partial x}(h_1 u_1^2 + \dfrac{1}{2} g h_1^2) = 0
\end{cases}
$$

SHALLOW FLOWS ON ARBITRARY SURFACE



Traditional boundary layer model

- No slip conditions for wind velocity on the solid surface
- Prandtl hypothesis: fast velocity gradients brings the balance of viscosity and inertia forces effects
- ***The viscous dumping is almost impossible***
  to neglect near surface even with high Reynolds number

Details of our boundary layer model

- To abandon no-slip conditions, horizontal velocity must be computing parameter
- To reduce order of the used equations
- To ensure high gradients of the flow near surface in consequence of Prandtl hypothesis
- ***To provide viscous dissipation by means of the scheme viscosity***

Model assumptions

- Volume concentration of the rigid particles is moderately high
- Viscosity and heat condition of the fluid and solid phases do not take effect at impulse and energy transfer in macroscopic scales
- Characteristic of the time scales of atmosphere motions are in excess of the interphase relaxation time

Governing Equations

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho v_x)}{\partial x} + \dfrac{\partial(\rho v_y)}{\partial y} + \dfrac{\partial(\rho v_z)}{\partial z} = 0 \\
\dfrac{\partial(\rho v_x)}{\partial t} + \dfrac{\partial(p + \rho v_x^2)}{\partial x} + \dfrac{\partial(\rho v_x v_y)}{\partial y} + \dfrac{\partial(\rho v_x v_z)}{\partial z} = 0 \\
\dfrac{\partial(\rho v_y)}{\partial t} + \dfrac{\partial(\rho v_x v_y)}{\partial x} + \dfrac{\partial(p + \rho v_y^2)}{\partial y} + \dfrac{\partial(\rho v_y v_z)}{\partial z} = 0 \\
\dfrac{\partial(\rho v_z)}{\partial t} + \dfrac{\partial(\rho v_x v_z)}{\partial x} + \dfrac{\partial(\rho v_y v_z)}{\partial y} + \dfrac{\partial(p + \rho v_z^2)}{\partial z} = 0 \\
\dfrac{\partial e}{\partial t} + \dfrac{\partial(e + p)v_x}{\partial x} + \dfrac{\partial(e + p)v_y}{\partial y} + \dfrac{\partial(e + p)v_z}{\partial z} = 0
\end{cases}
$$

$$e = \rho(\varepsilon + \frac{v_x^2 + v_y^2 + v_z^2}{2})$$

ρ – density,

p – pressure,

v_x, v_y, v_z – velocity,

e – total energy per unit volume,

ε – internal energy per unit volume

Model physics

Model of the effective perfect gas for atmosphere with solid particles

$$\varepsilon = cT, \quad p = \rho RT$$

$$R = x_1 R_1, \quad c = x_1 c_1 + x_2 c_2$$

$$x_i = \frac{\rho_i}{\rho}, \quad x_1 + x_2 = 1$$

R, R_1 – gas constants for mixture and fluid phase

c, c_1, c_2 – thermal capacities for mixture and each phase

ρ_i – density of each phase

Advantages of the atmosphere model

Equations for dusty atmosphere are similar perfect fluid equations with changed specific heat ratio γ and sound speed Cs

$$\gamma = (c + R)/c \le \gamma_1$$

$$C_s = \sqrt{\frac{\gamma p}{\rho}} = C_{1s}\sqrt{\frac{\gamma x_1}{\gamma_1}} \le C_{1s}$$

Possibility to describe uniformly atmosphere flows over complex terrain in the conditions of impurity lifting and deposition

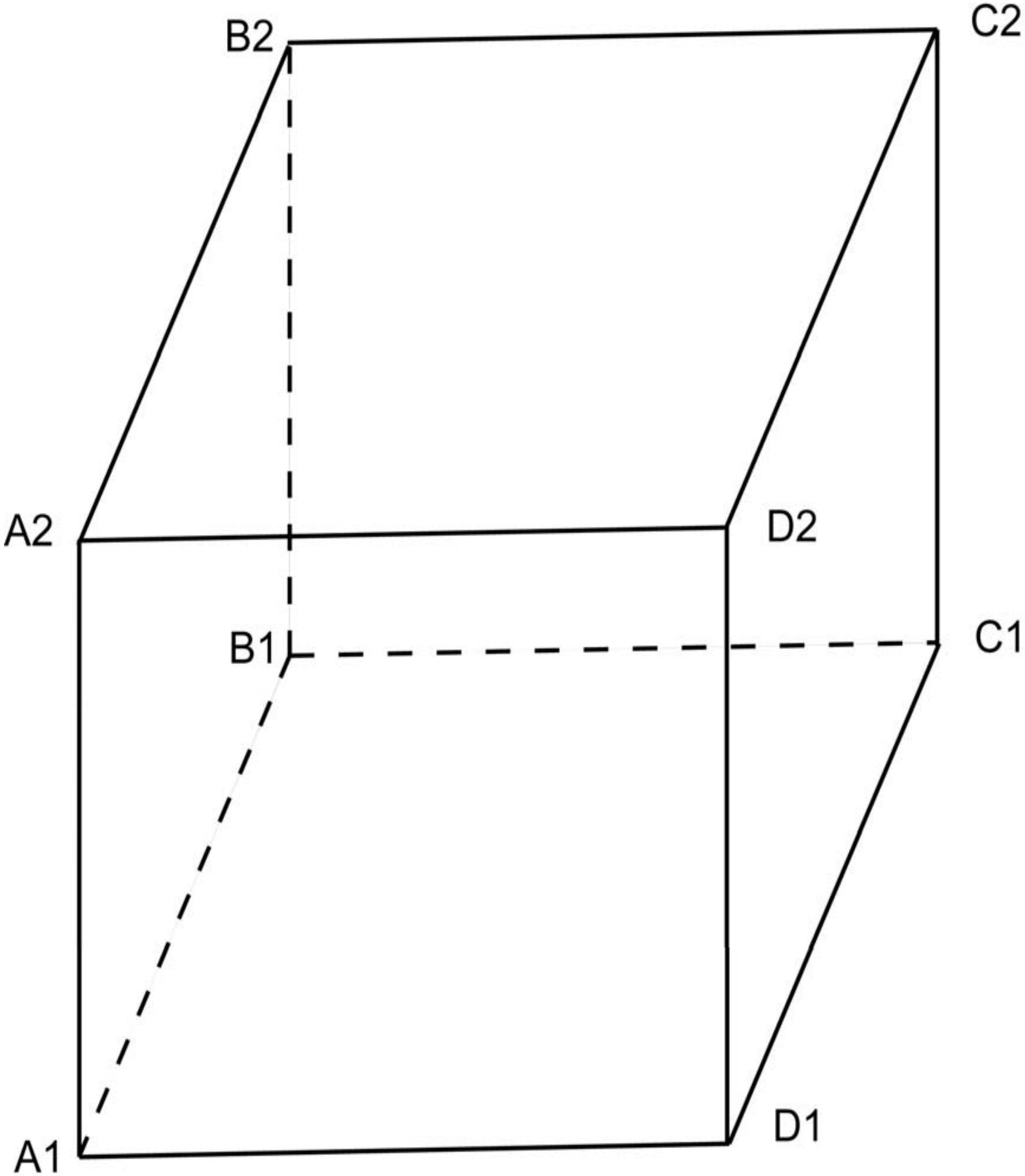
Integral form of the equations

$$\begin{cases}
\iiint\limits_L \dfrac{\partial \rho}{\partial t}\,dtdxdy + \iiint\limits_L \dfrac{\partial \rho u}{\partial x}\,dtdxdy + \iiint\limits_L \dfrac{\partial \rho v}{\partial y}\,dtdxdy = 0 \\
\iiint\limits_L \dfrac{\partial \rho u}{\partial t}\,dtdxdy + \iiint\limits_L \dfrac{\partial (p + \rho u^2)}{\partial x}\,dtdxdy + \iiint\limits_L \dfrac{\partial \rho uv}{\partial y}\,dtdxdy = 0 \\
\iiint\limits_L \dfrac{\partial \rho v}{\partial t}\,dtdxdy + \iiint\limits_L \dfrac{\partial \rho uv}{\partial x}\,dtdxdy + \iiint\limits_L \dfrac{\partial (p + \rho v^2)}{\partial y}\,dtdxdy = 0 \\
\iiint\limits_L \dfrac{\partial e}{\partial t}\,dtdxdy + \iiint\limits_L \dfrac{\partial (e + p)u}{\partial x}\,dtdxdy + \iiint\limits_L \dfrac{\partial (e + p)v}{\partial y}\,dtdxdy = 0
\end{cases}$$

L-arbitrary volume

Integral form of the equations after using Gauss theorem

$$
\begin{cases}
\oiint_{\partial L} \rho dxdy + \rho udtdy + \rho vdtdx = 0 \\
\oiint_{\partial L} \rho udxdy + (p + \rho u^2)udtdy + \rho uvdtdx = 0 \\
\oiint_{\partial L} \rho vdxdy + \rho uvdtdy + (p + \rho v^2)dtdx = 0 \\
\oiint_{\partial L} edxdy + (e + p)udtdy + (e + p)vdtdx = 0
\end{cases}
$$


B2
C2
A2
D2
B1
C1
A1
D1

Computational algorithm

$$R_1 = R_1 + \frac{\tau}{S_{ABCD}}(R_1^{(1)}u^{(1)}\sin\alpha\,|AB| - R_1^{(3)}u^{(3)}\sin\beta\,|CD| + R_1^{(4)}v^{(4)}\,|AD| - R_1^{(2)}v^{(2)}\,|BC| + R_1^{(1)}v^{(1)}\cos\alpha\,|AB| - R_1^{(3)}v^{(3)}\cos\beta\,|CD|)$$

$$R_2 = R_2 + \frac{\tau}{S_{ABCD}}(R_2^{(1)}u^{(1)}\sin\alpha\,|AB| - R_2^{(3)}u^{(3)}\sin\beta\,|CD| + R_2^{(4)}v^{(4)}\,|AD| - R_2^{(2)}v^{(2)}\,|BC| + R_2^{(1)}v^{(1)}\cos\alpha\,|AB| - R_2^{(3)}v^{(3)}\cos\beta\,|CD|)$$

$$R_1 u = R_1 u + \frac{\tau}{S_{ABCD}}((P^{(1)} + R_1^{(1)}u^{(1)^2})\,|AB|\sin\alpha - (P^{(3)} + R_1^{(3)}u^{(3)^2})\,|CD|\sin\beta + R_1^{(4)}u^{(4)}v^{(4)}\,|AD| - R_1^{(2)}u^{(2)}v^{(2)}\,|BC| +$$
$$R_1^{(1)}u^{(1)}v^{(1)}\,|AB|\cos\alpha - R_1^{(3)}u^{(3)}v^{(3)}\,|CD|\cos\beta)$$

$$R_1 v = R_1 v + \frac{\tau}{S_{ABCD}}((P^{(1)} + R_1^{(1)}v^{(1)^2})\,|AB|\cos\alpha - (P^{(2)} + R_1^{(2)}v^{(2)^2})\,|BC| - (P^{(3)} + R_1^{(3)}v^{(3)^2})\,|CD|\cos\beta +$$
$$(P^{(4)} + R_1^{(4)}v^{(4)^2})\,|AD| + R_1^{(1)}u^{(1)}v^{(1)}\,|AB|\sin\alpha - R_1^{(3)}u^{(3)}v^{(3)}\,|CD|\sin\beta)$$

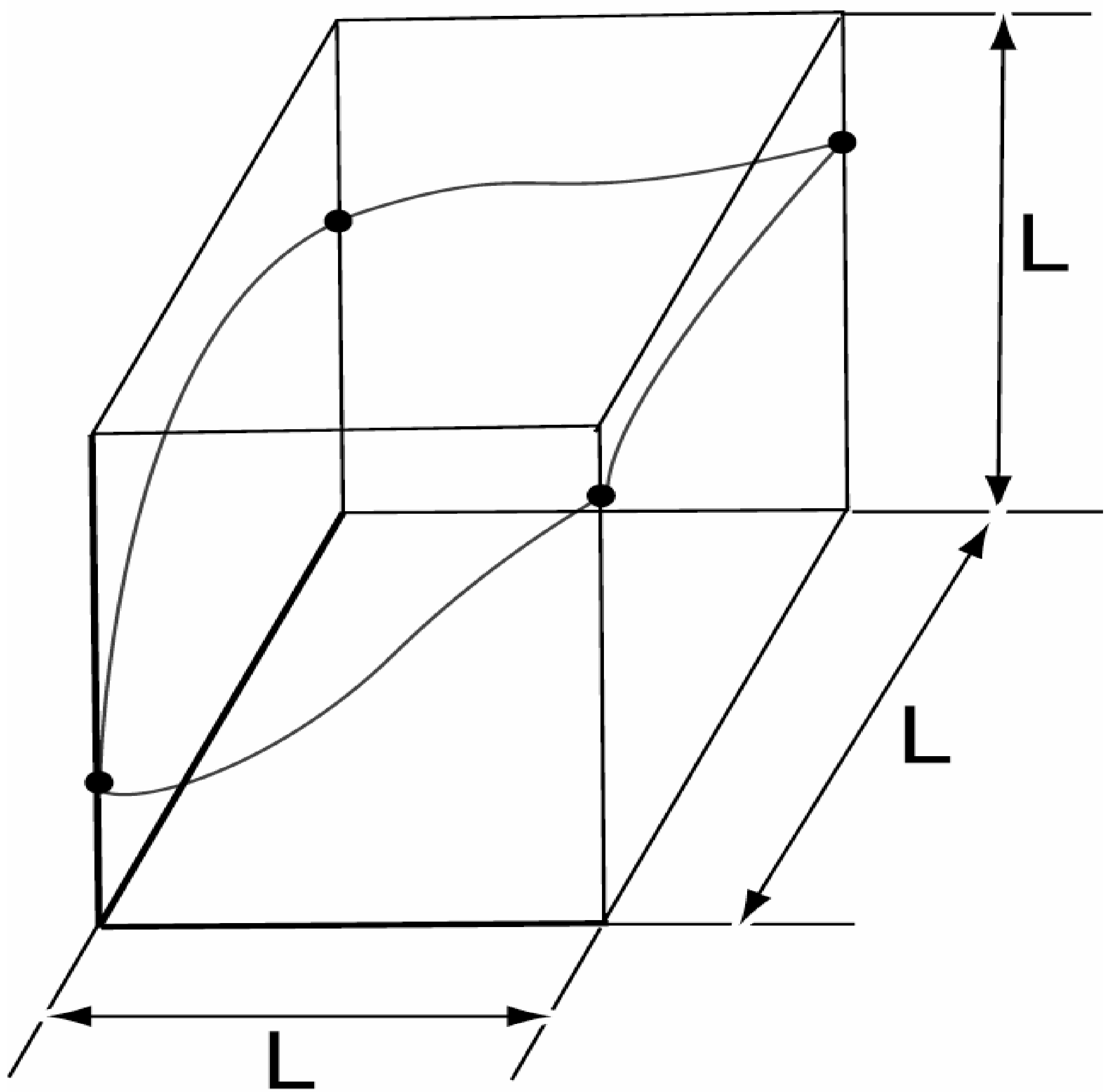
$$E = E + \frac{\tau}{S_{ABCD}}((E^{(1)} + P^{(1)})v^{(1)}\,|AB|\sin\alpha - (E^{(3)} + P^{(3)})u^{(3)}\,|CD|\sin\beta + (E^{(4)} + P^{(4)})v^{(4)}\,|AD| - (E^{(2)} + E^{(2)})v^{(2)}\,|BC| +$$
$$(E^{(1)} + P^{(1)})v^{(1)}\,|AB|\cos\alpha - (E^{(3)} + P^{(3)})v^{(3)}\,|CD|\cos\beta)$$

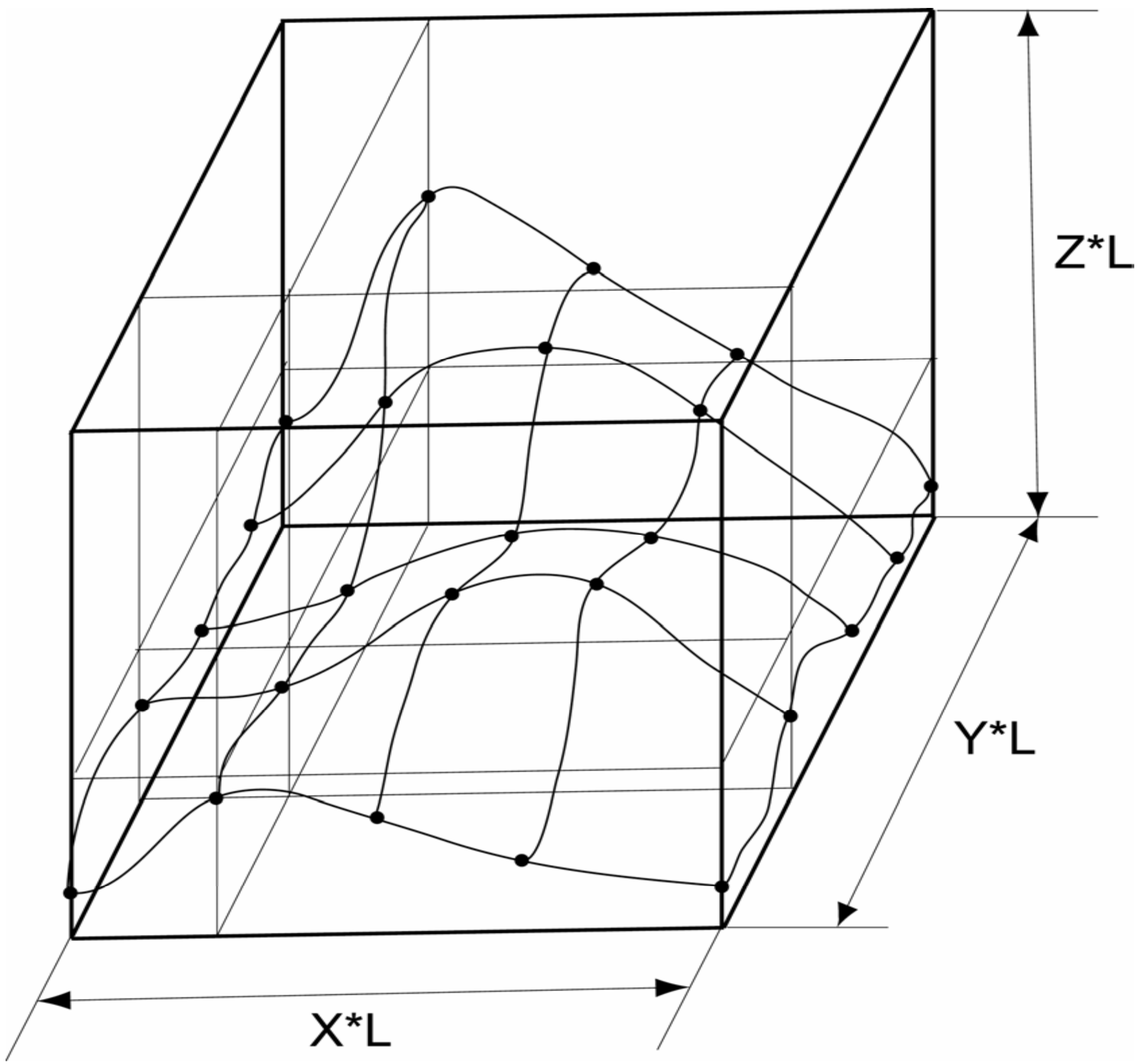
Details of Godunov method

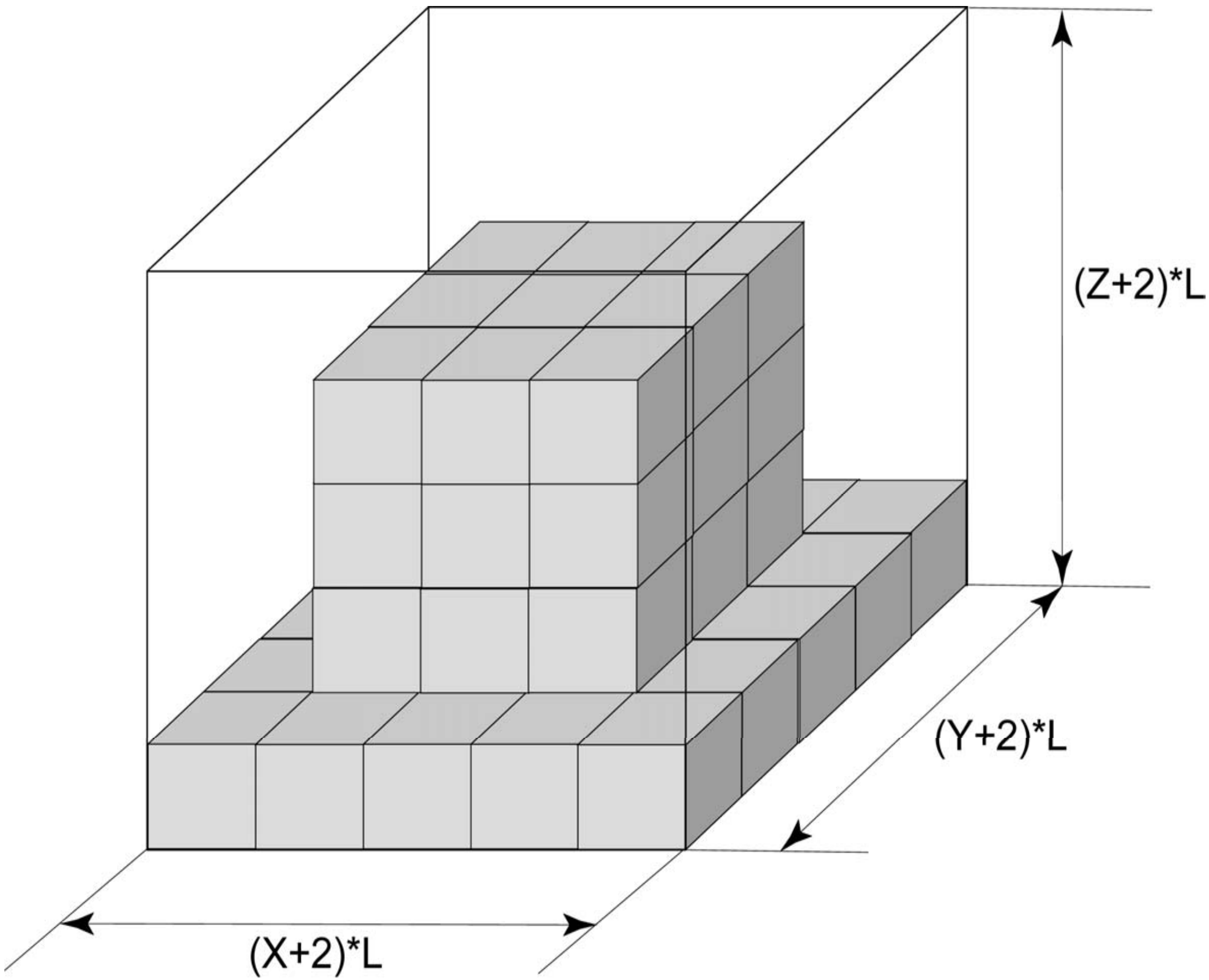
- Divergent difference scheme
- Provides for the viscosity by local tangential gaps at every step of the grid
- Such tangential gaps do not manifest themselves on the outer flow scale and make available dissipation energy

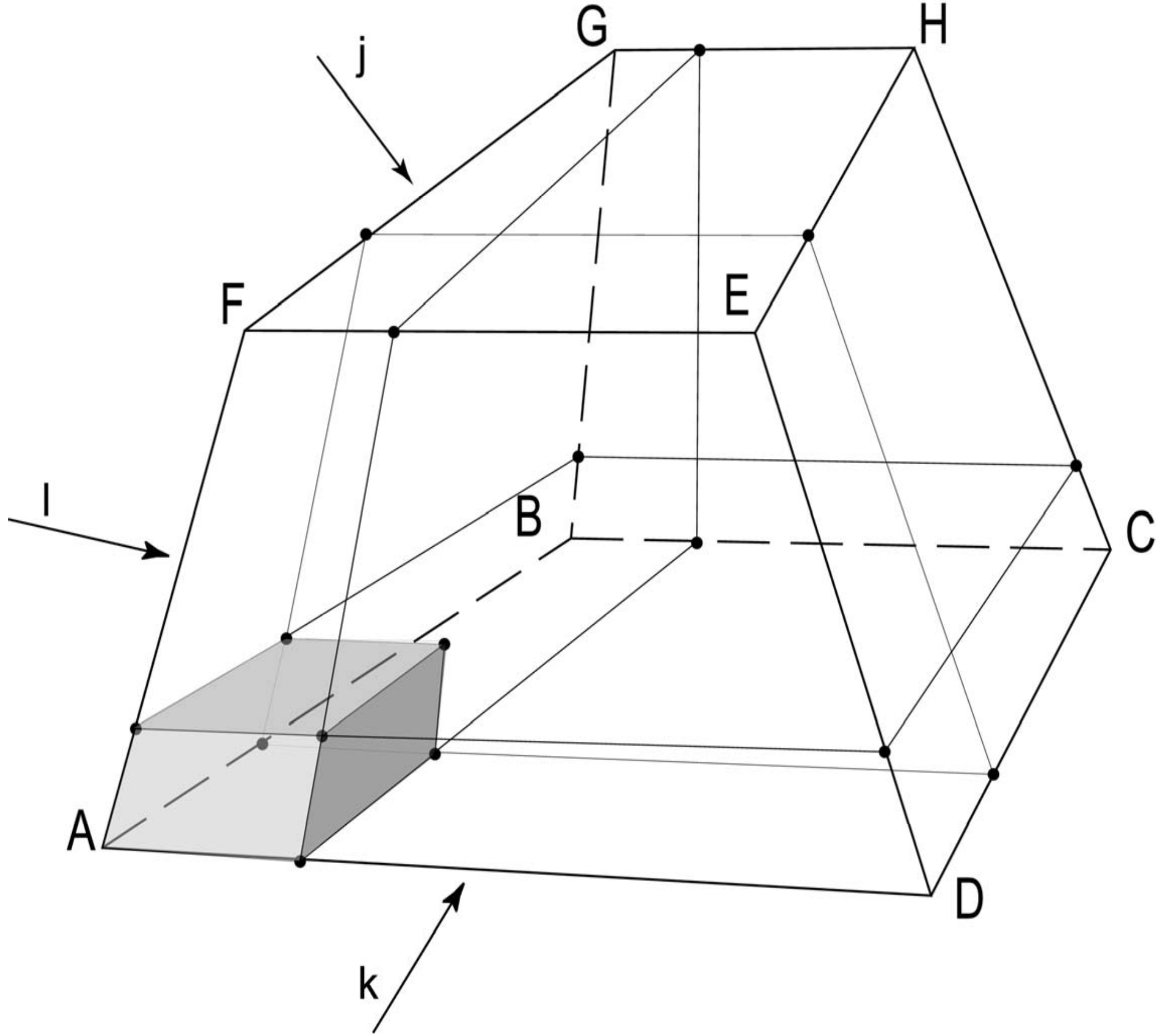
Digitization procedure

- Computational domain is sectored on curved trapezium grid by two set of lines
- Transformed computational space consists of rectangles
- If each grid hydrodynamic quantities are replaced by certain average constant at present instants of time
- Beginning with initial constant in each grid we make computations step by step at time intervals of interest to us

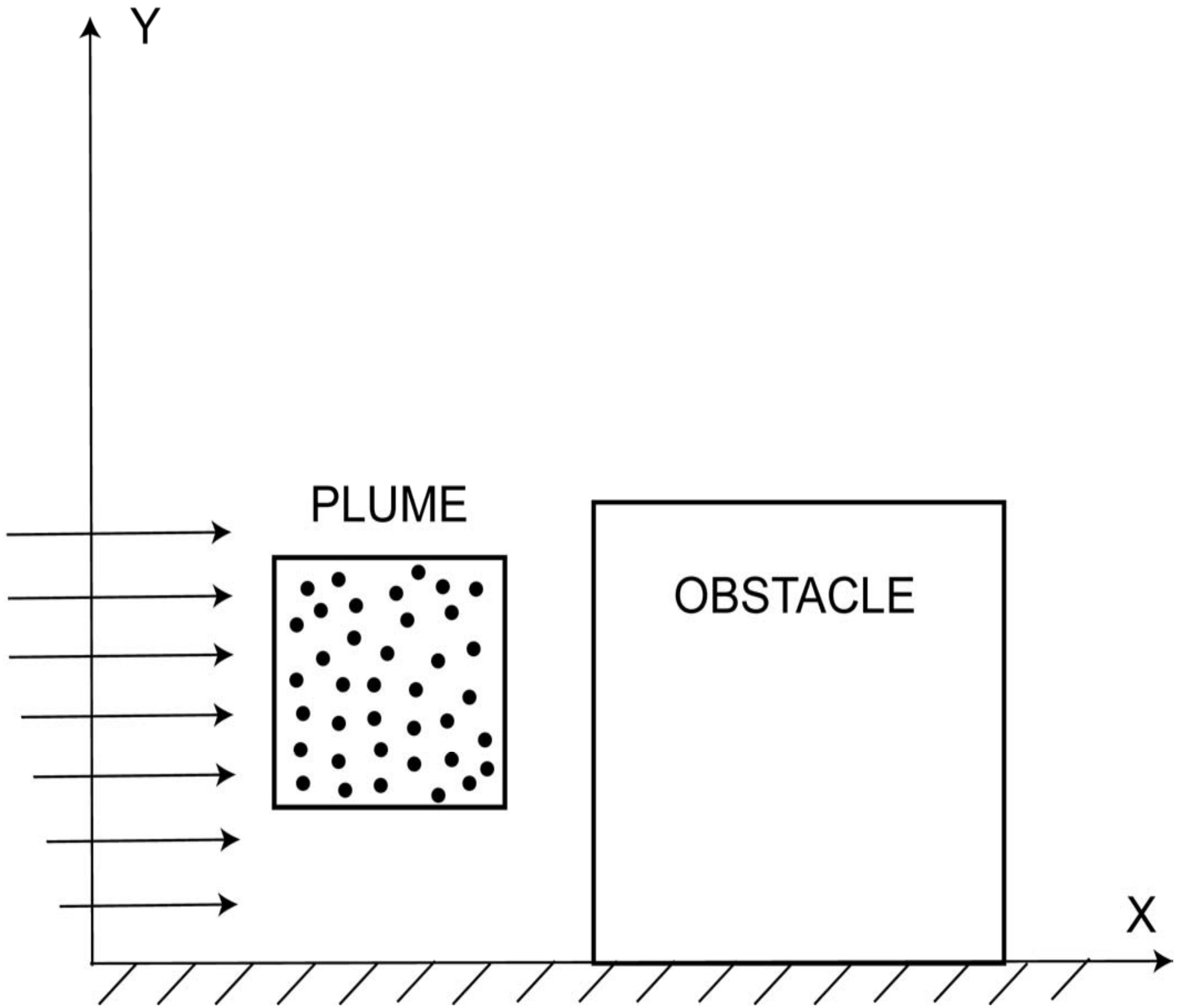

L
L
L

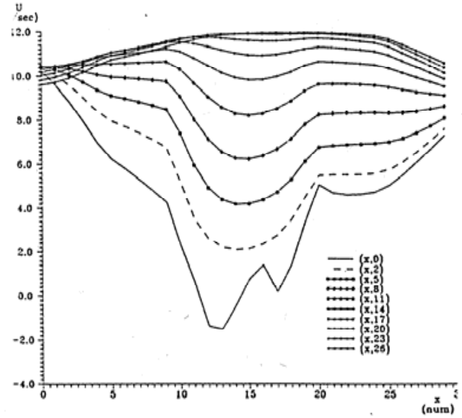

Z*L
Y*L
X*L

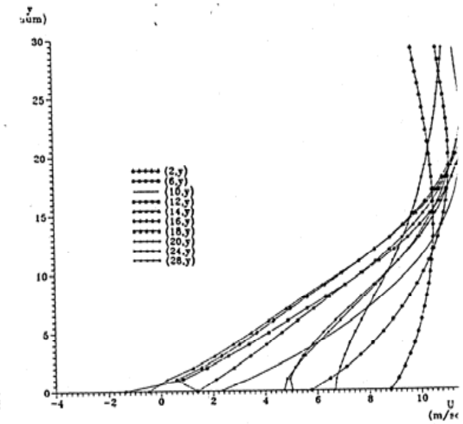

(Z+2)*L
(Y+2)*L
(X+2)*L

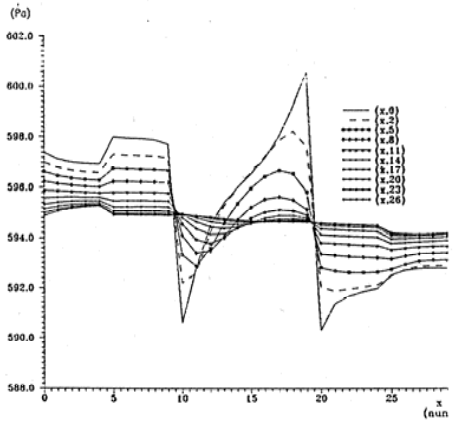


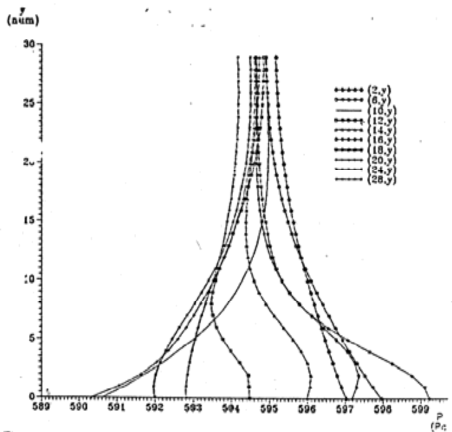
G
H
j
F
E
I
B
C
A
D
k

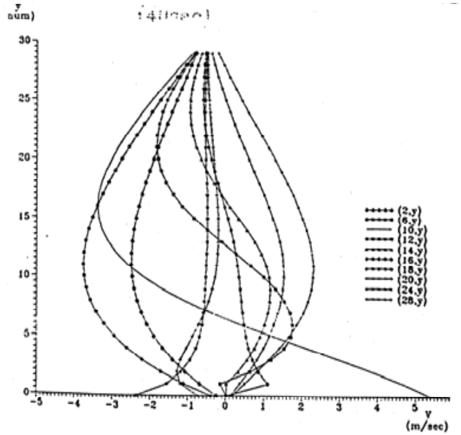

Y
PLUME
OBSTACLE
X

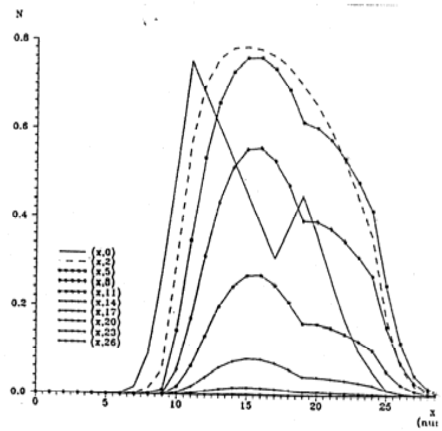

U
(m/sec)
12.0
10.0
8.0
6.0
4.0
2.0
0.0
-2.0
-4.0
0
5
10
15
20
25
x
(num)
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)

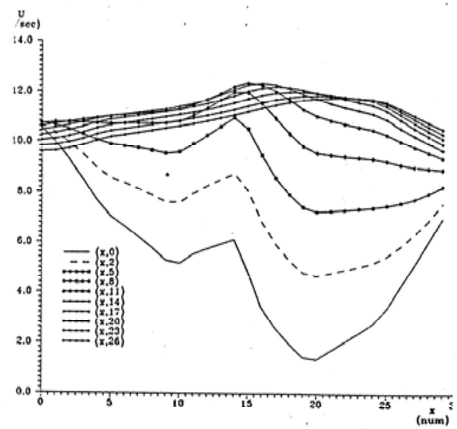

y
(num)
30
25
20
15
10
5
0
-4
-2
0
2
4
6
8
10
U
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)

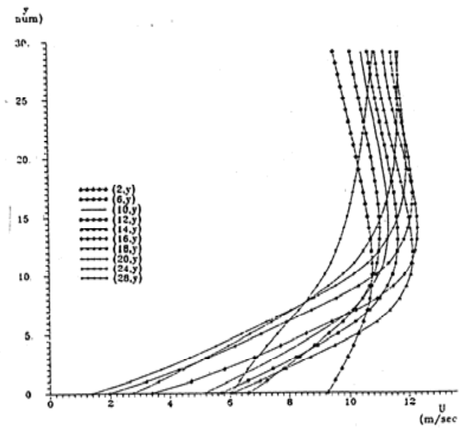

(Pa)
602.0
600.0
598.0
596.0
594.0
592.0
590.0
588.0
0
5
10
15
20
25
x
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)

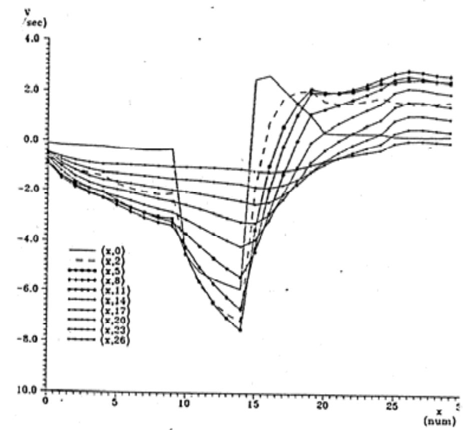

y
(num)
30
25
15
10
5
0
589
590
591
592
593
594
595
596
597
598
599
P
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)

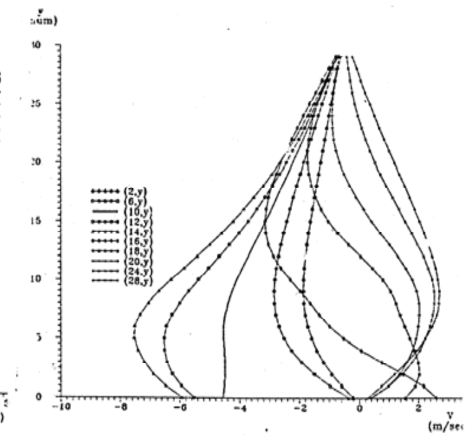

y
(num)
30
25
20
15
10
5
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
V
(m/sec)
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)

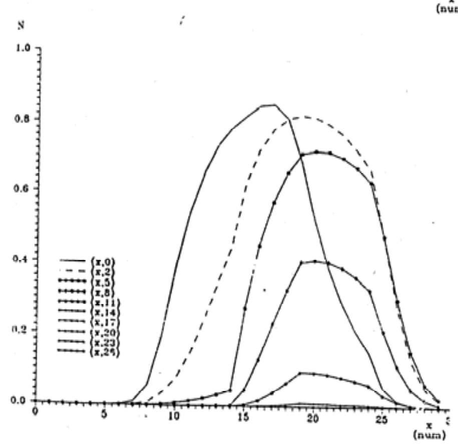

N
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
x
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)

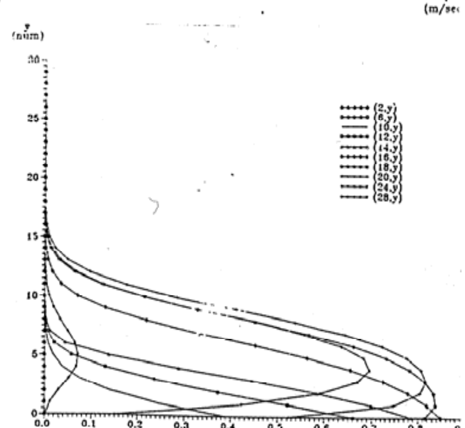

U
/sec)
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)
x
(num)

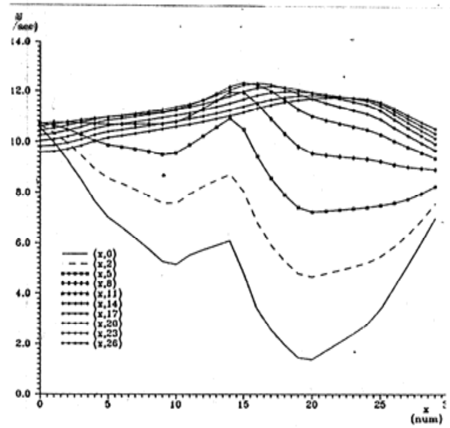

y
num)
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)
U
(m/sec


V
/sec)
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)
x
(num)

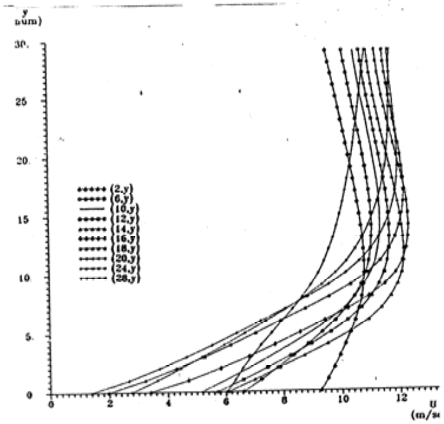

y
num)
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)
V
(m/sec


N
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)
x
(num)

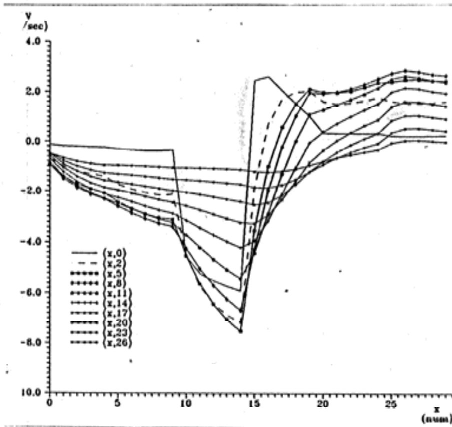

y
(num)
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)
N



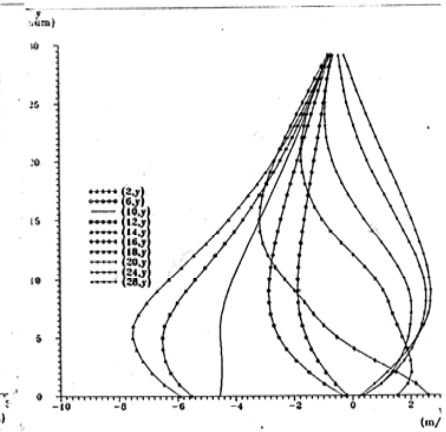
/sec)
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)
x
(num)



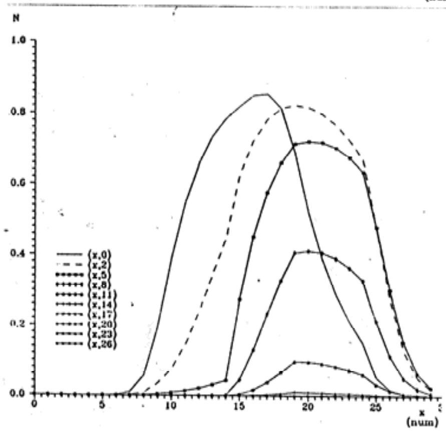
y
(num)
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)
U



V
/sec)
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
(x,23)
(x,26)
x
(num)



y
(num)
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)



N
(x,0)
(x,2)
(x,5)
(x,8)
(x,11)
(x,14)
(x,17)
(x,20)
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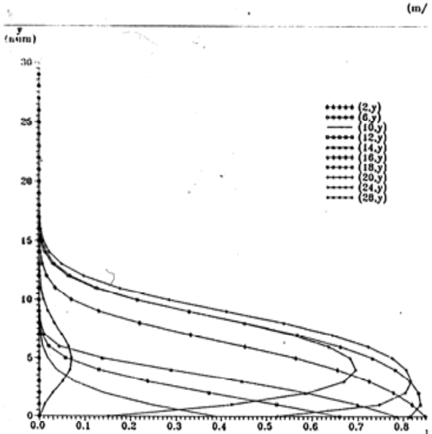
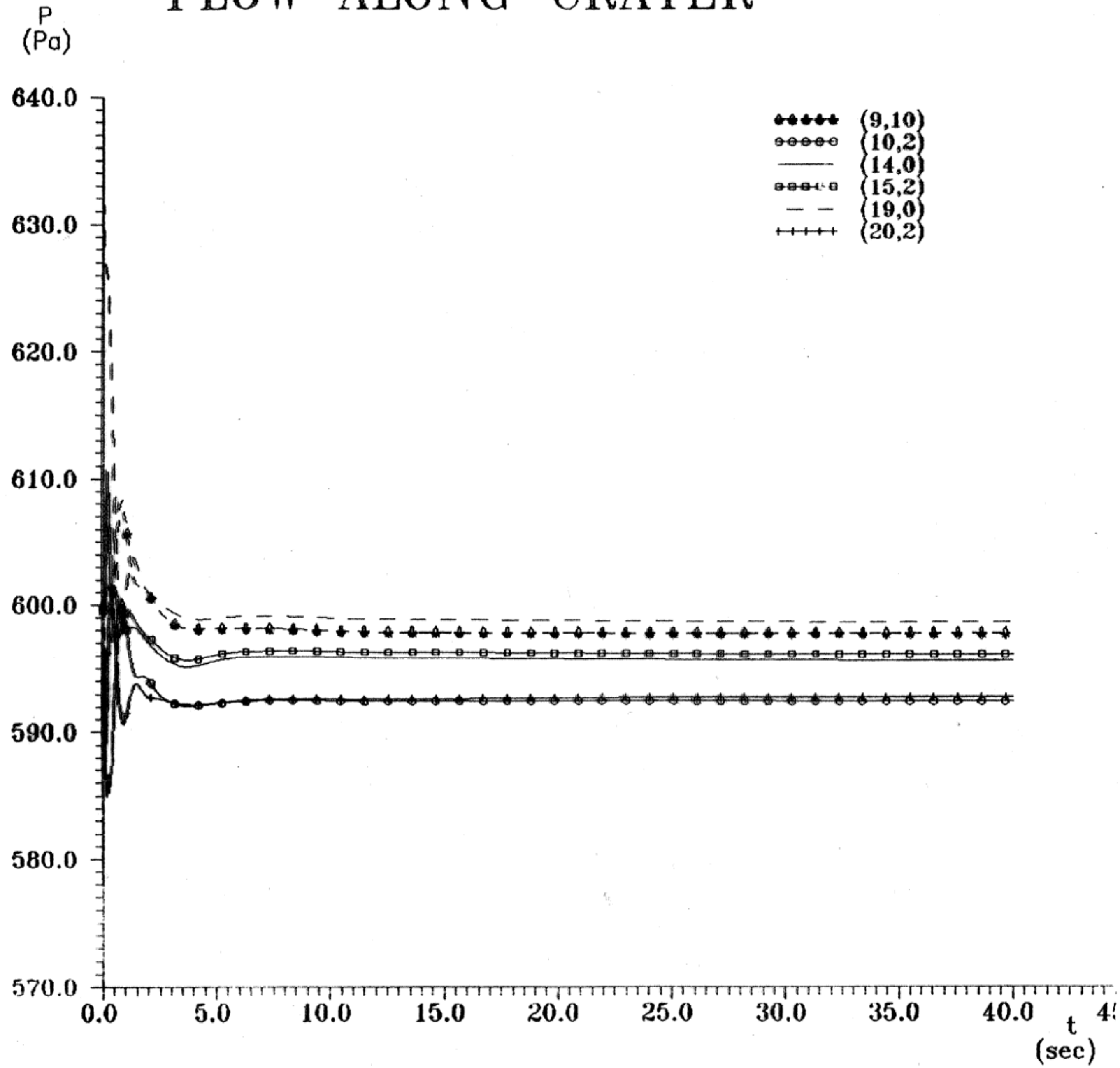
(x,26)
x
(num)

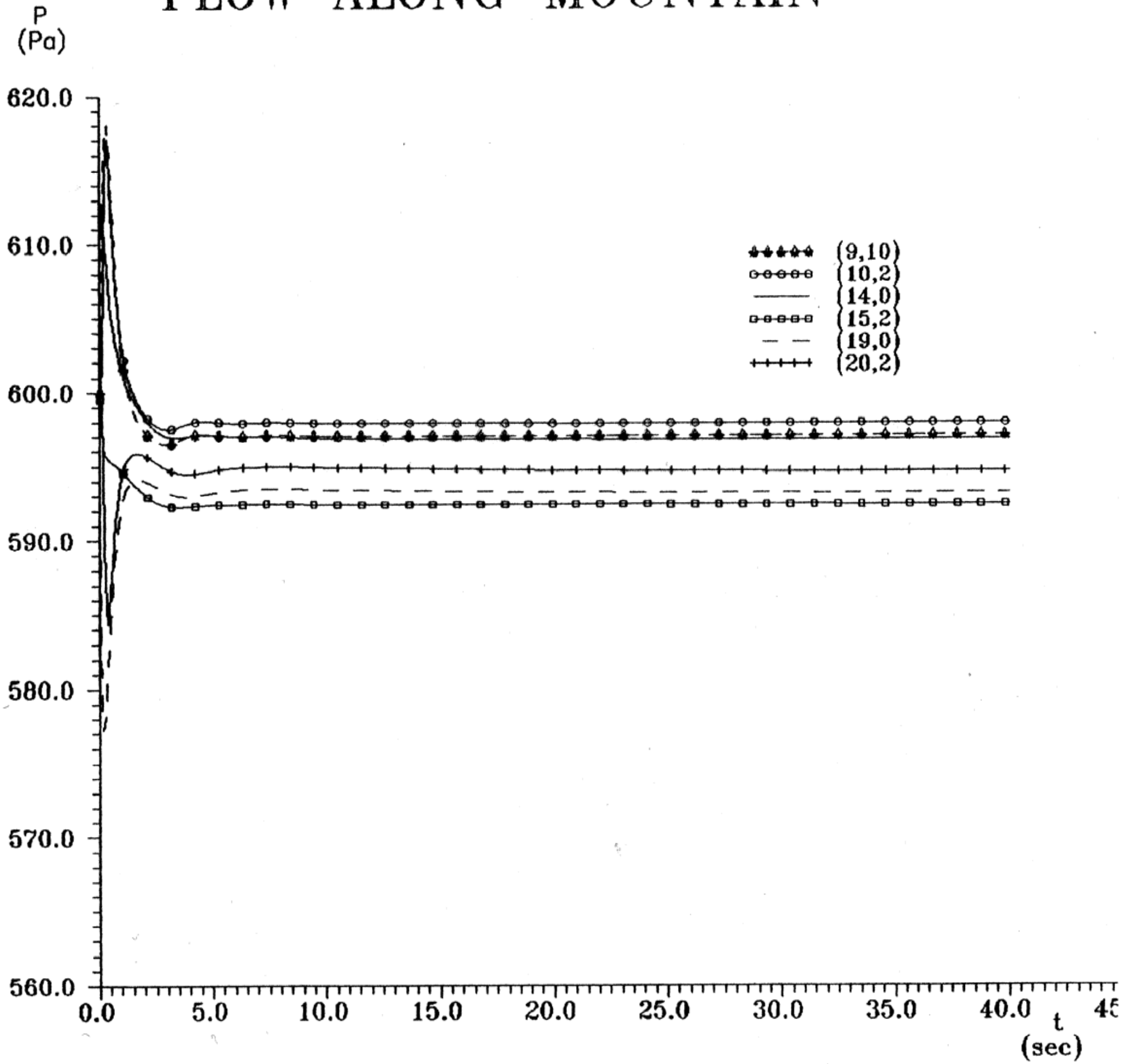


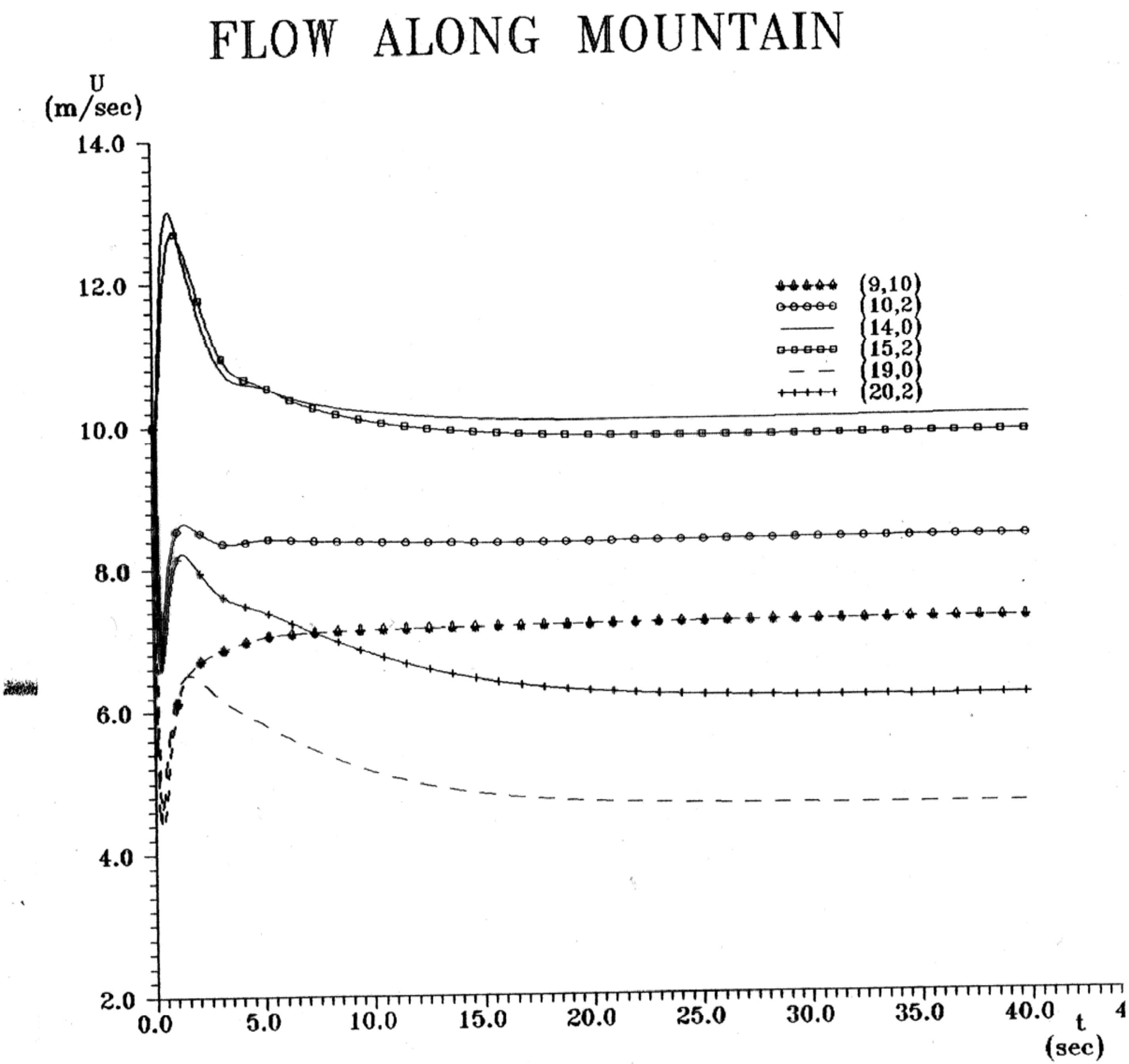
y
(num)
(2,y)
(6,y)
(10,y)
(12,y)
(14,y)
(16,y)
(18,y)
(20,y)
(24,y)
(28,y)

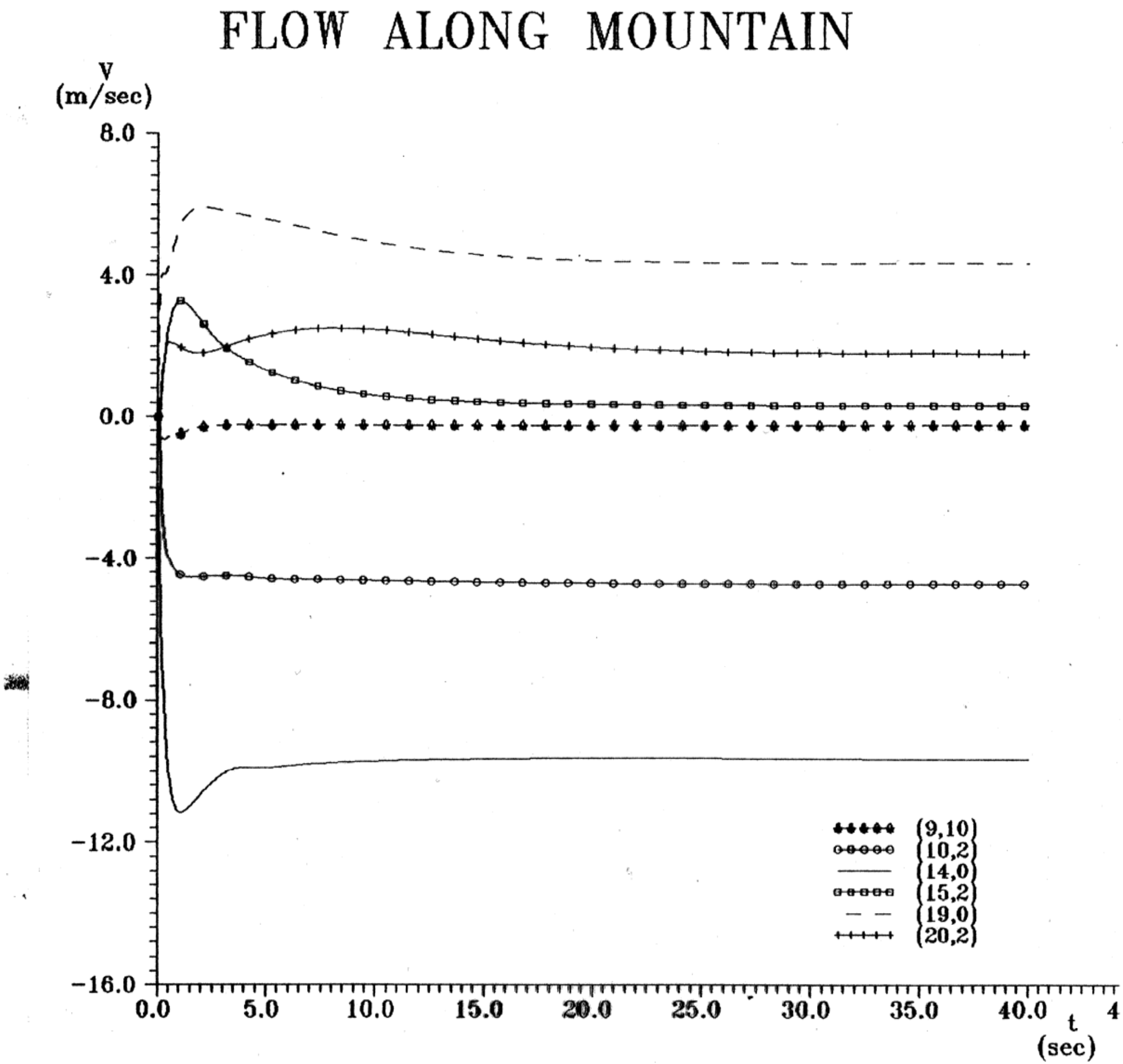
FLOW ALONG CRATER



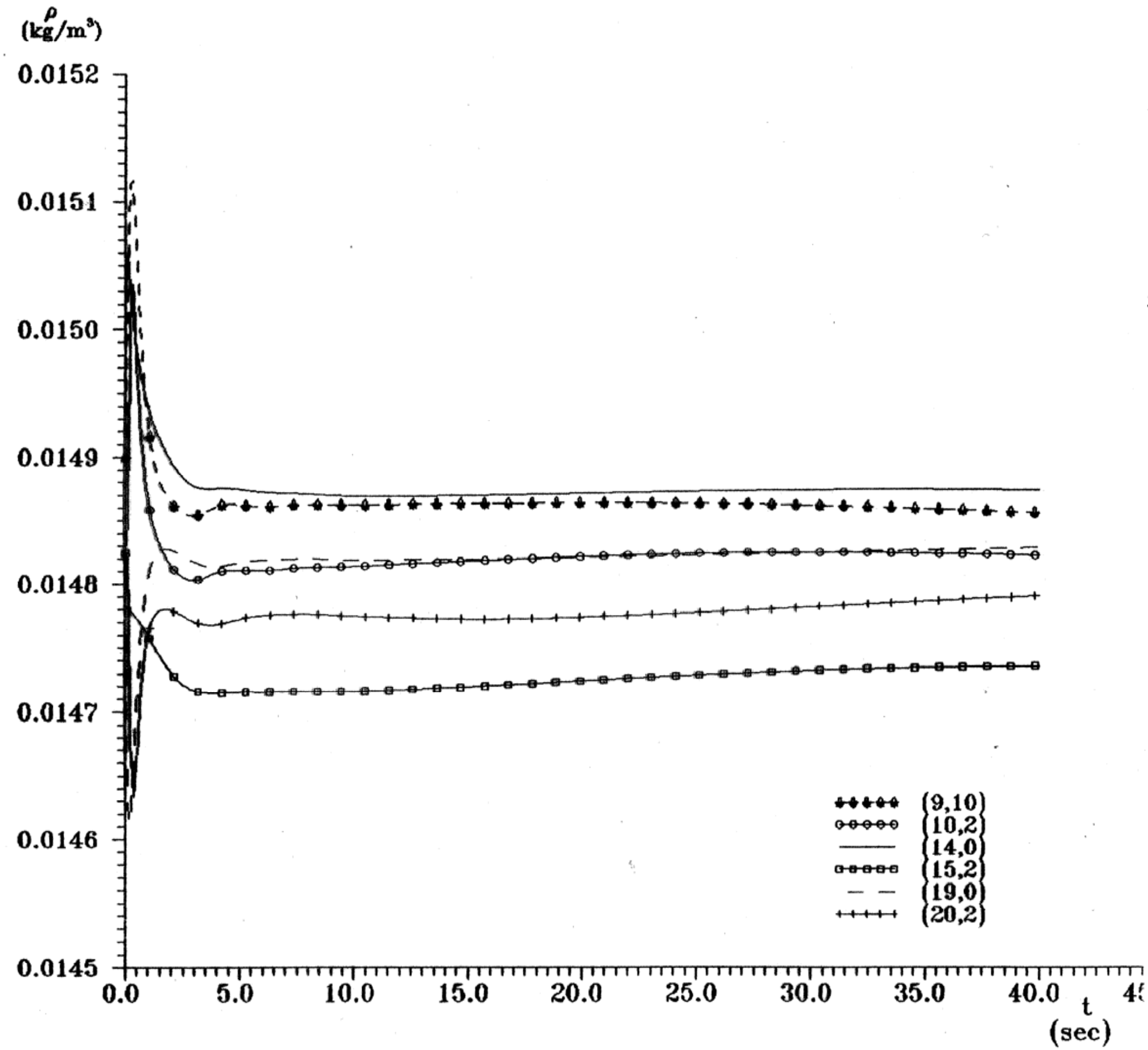
FLOW ALONG MOUNTAIN

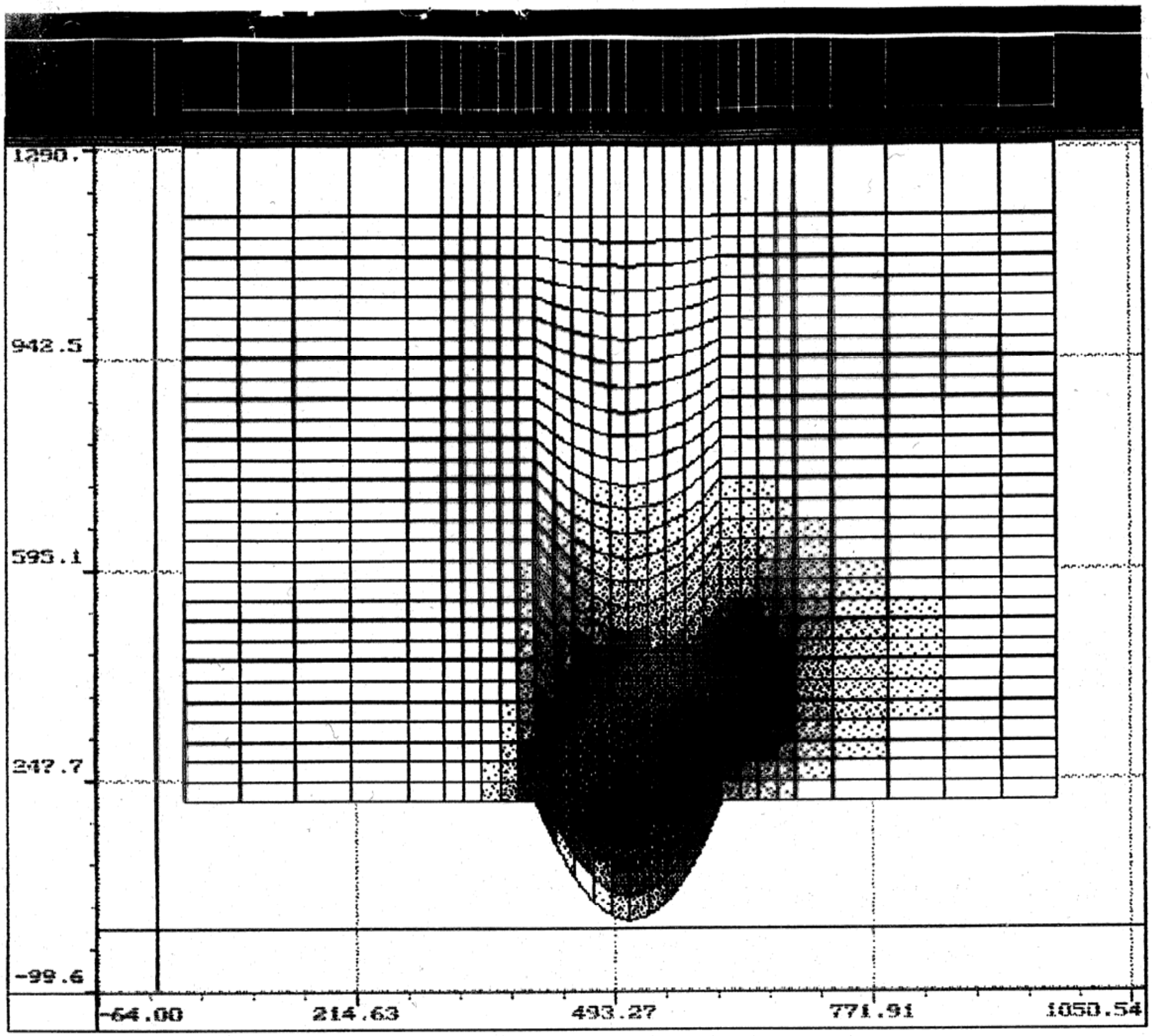



FLOW ALONG MOUNTAIN
U
(m/sec)
14.0
12.0
10.0
8.0
6.0
4.0
2.0
0.0
5.0
10.0
15.0
20.0
25.0
30.0
35.0
40.0
t
(sec)
(9,10)
(10,2)
(14,0)
(15,2)
(19,0)
(20,2)

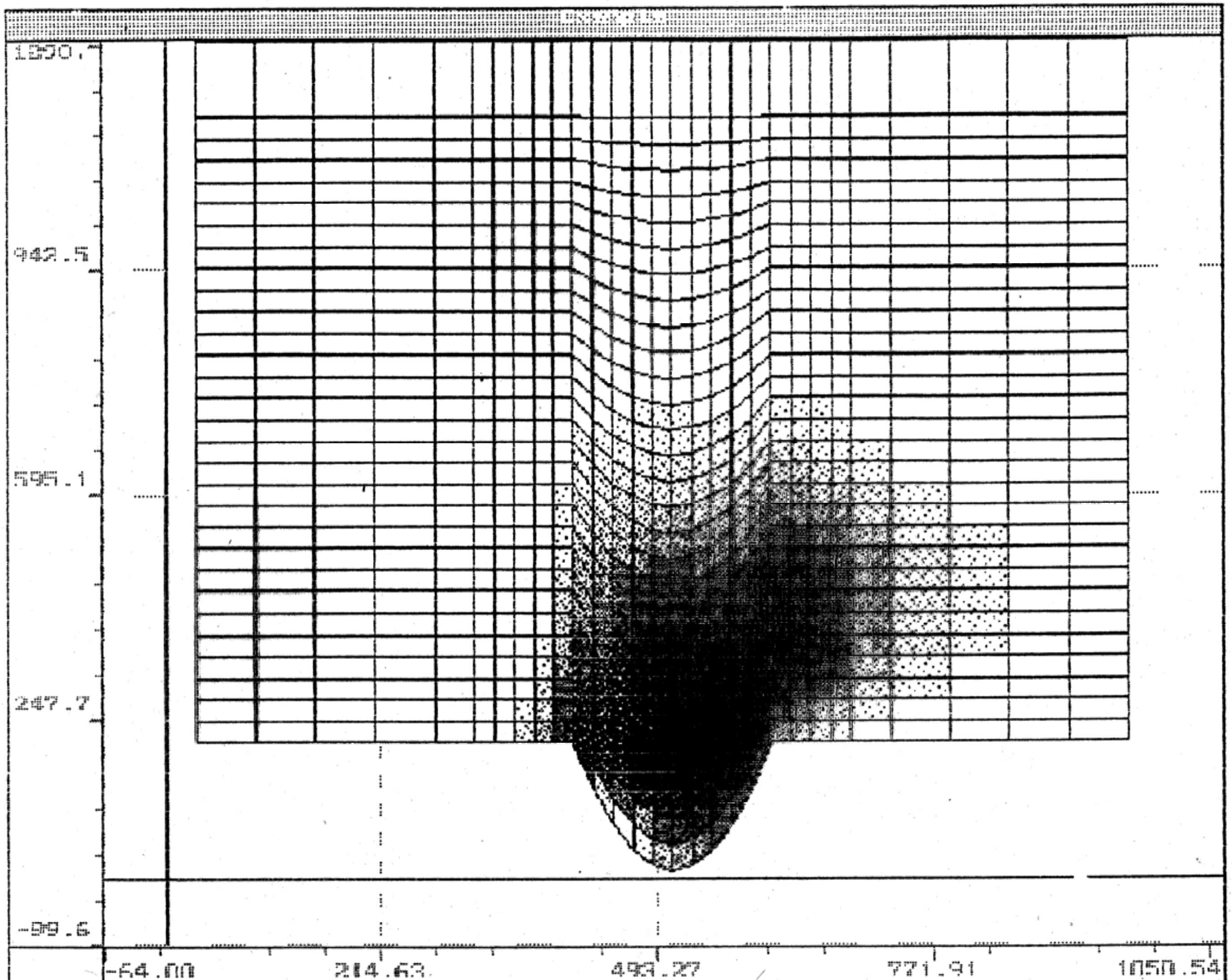

FLOW ALONG MOUNTAIN
V
(m/sec)
8.0
4.0
0.0
-4.0
-8.0
-12.0
-16.0
0.0
5.0
10.0
15.0
20.0
25.0
30.0
35.0
40.0
4
t
(sec)
(9,10)
(10,2)
(14,0)
(15,2)
(19,0)
(20,2)

FLOW ALONG MOUNTAIN

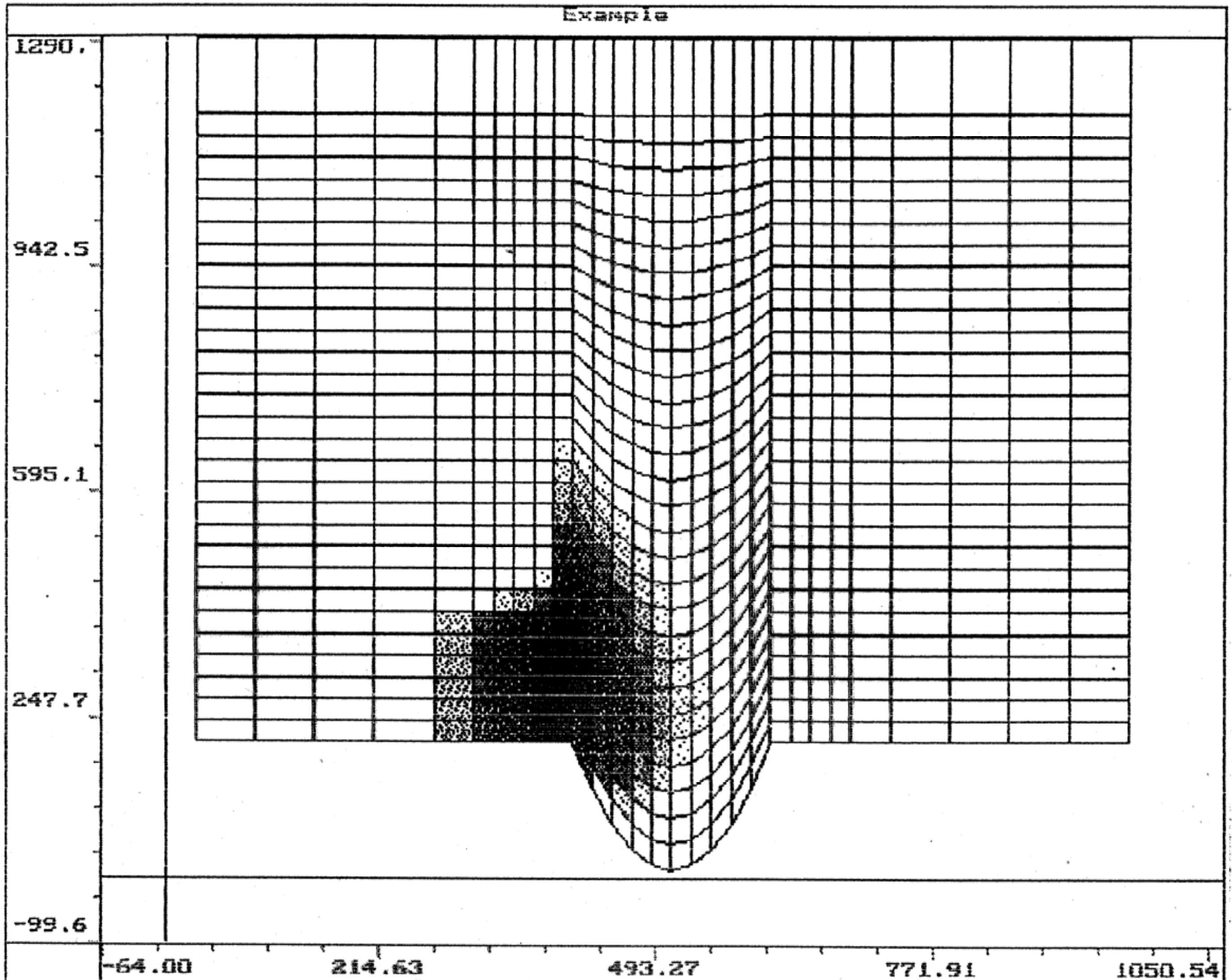


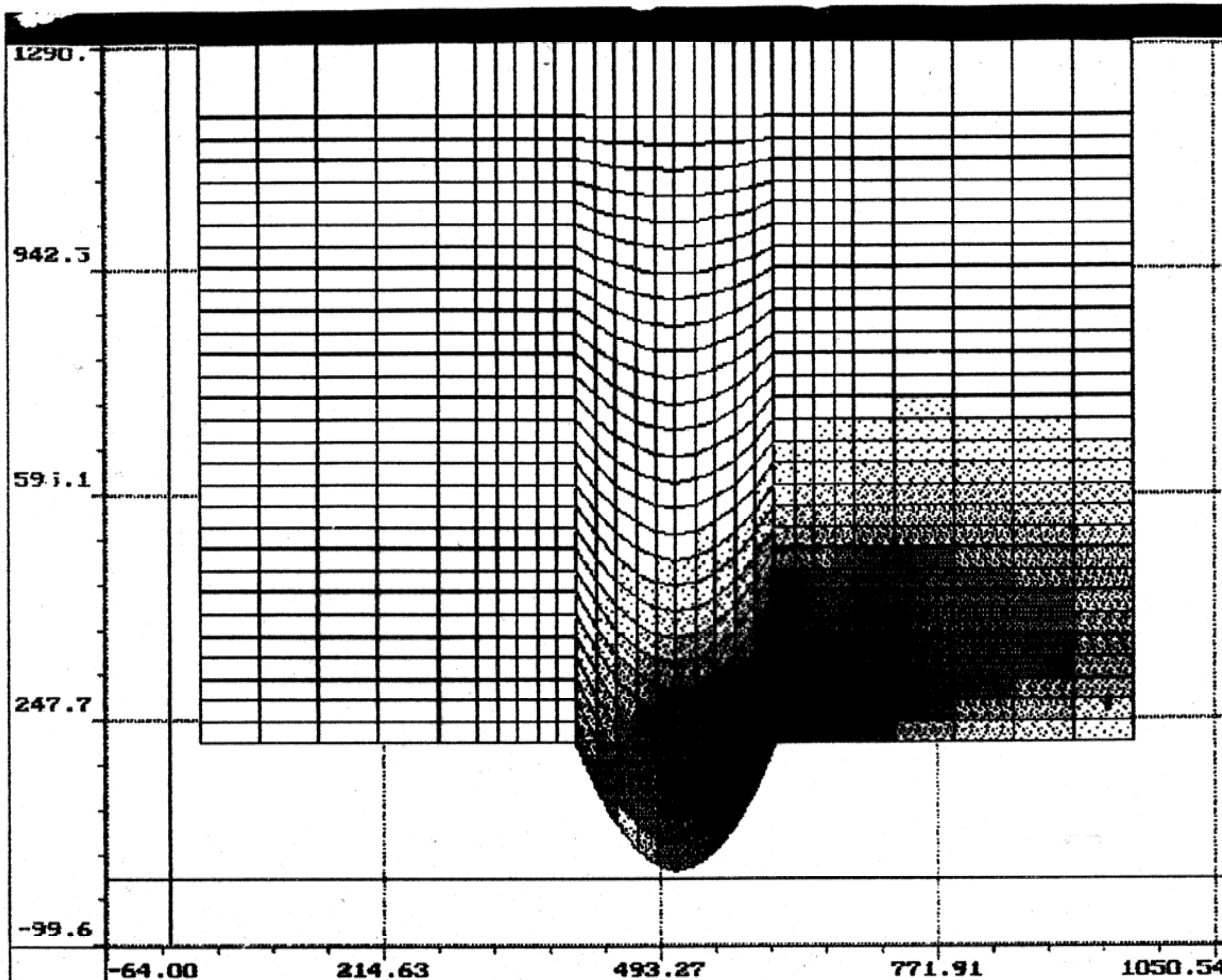


1290.
942.5
595.1
247.7
-99.6
-64.00
214.63
493.27
771.91
1050.54



1290.7
942.5
595.1
247.7
-99.6
-64.00
214.63
493.27
771.91
1050.54


Example
1290.
942.5
595.1
247.7
-99.6
-64.00
214.63
493.27
771.91
1050.54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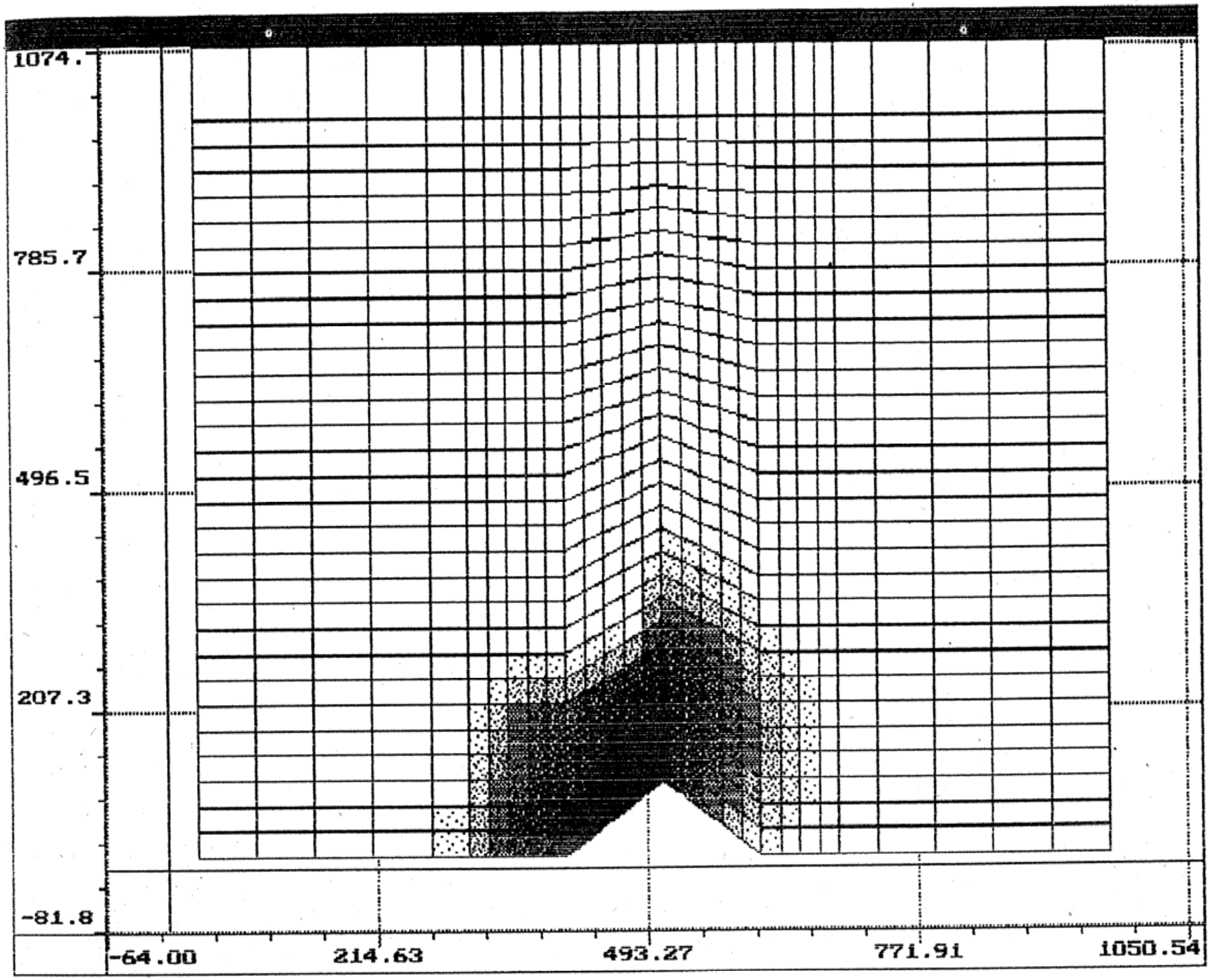
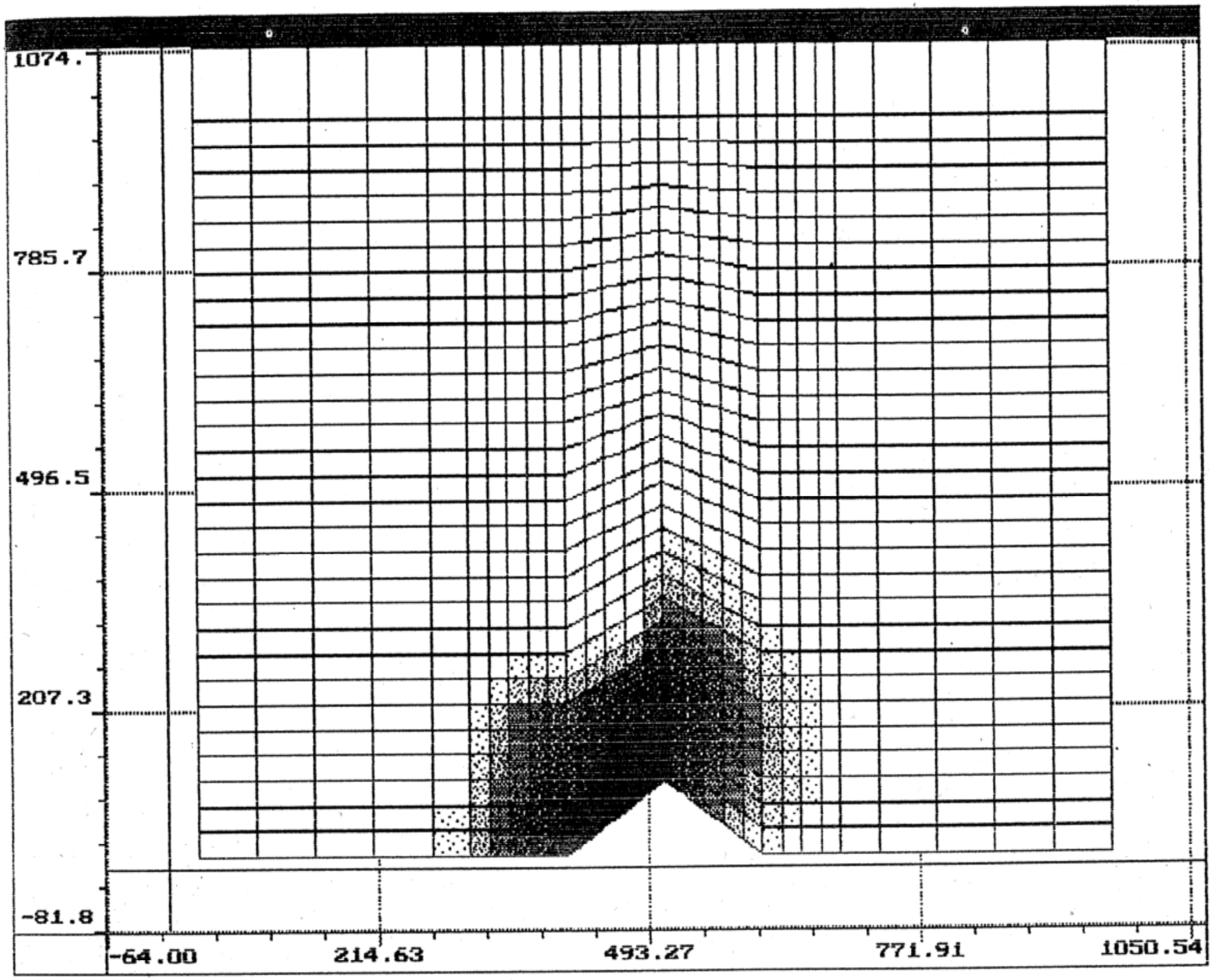
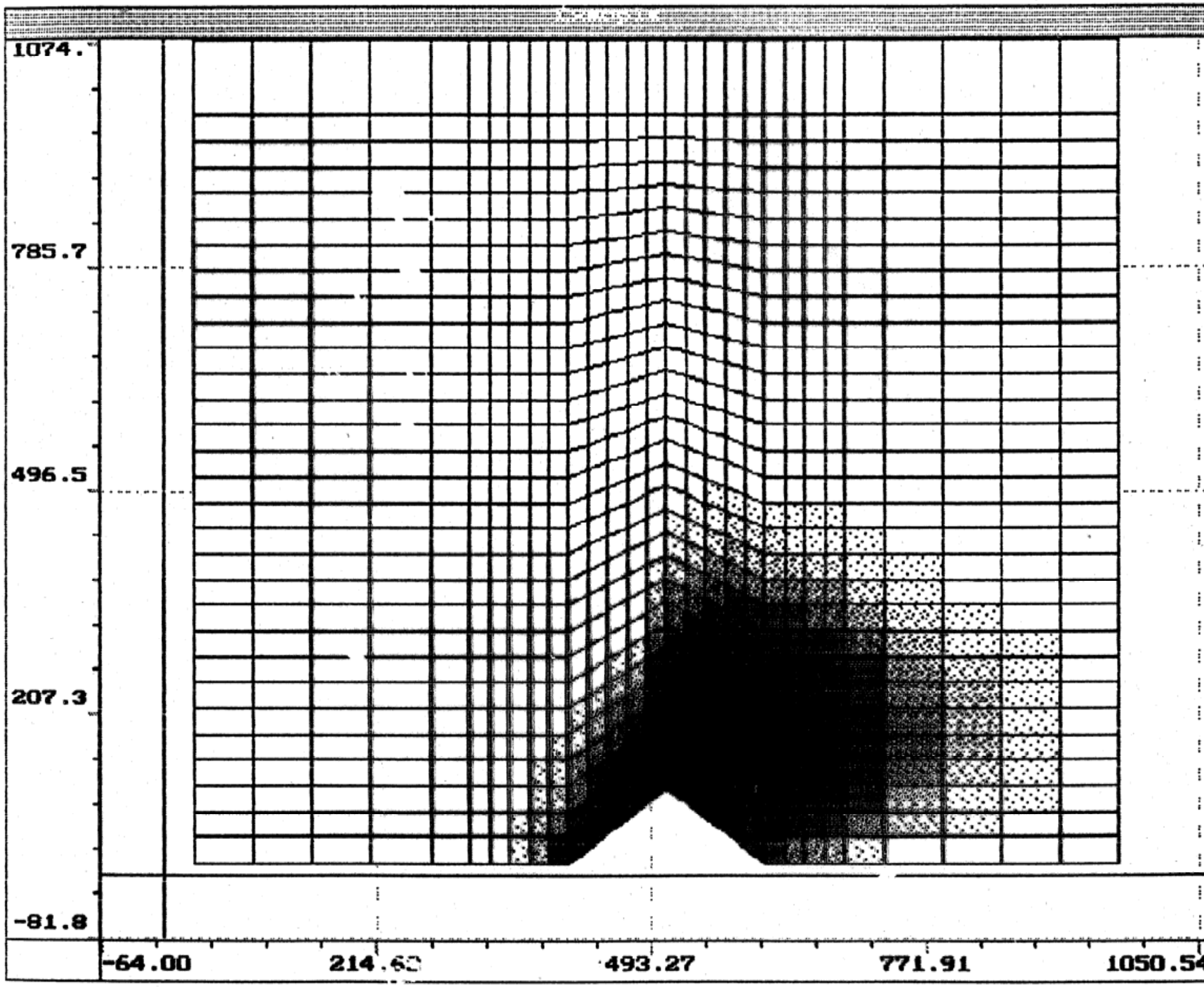
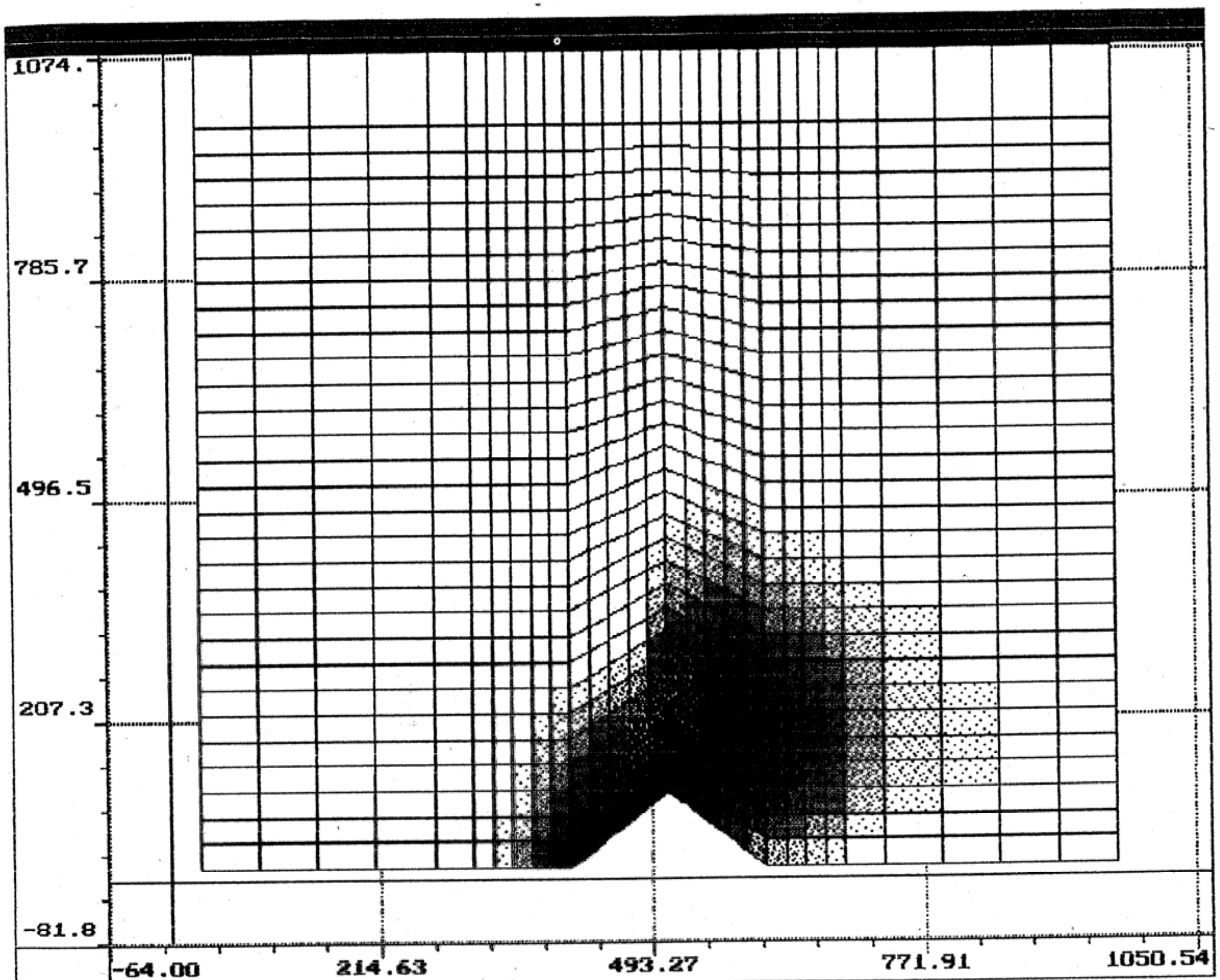
942.3
595.1
247.7
-99.6
-64.00
214.63
493.27
771.91
1050.54



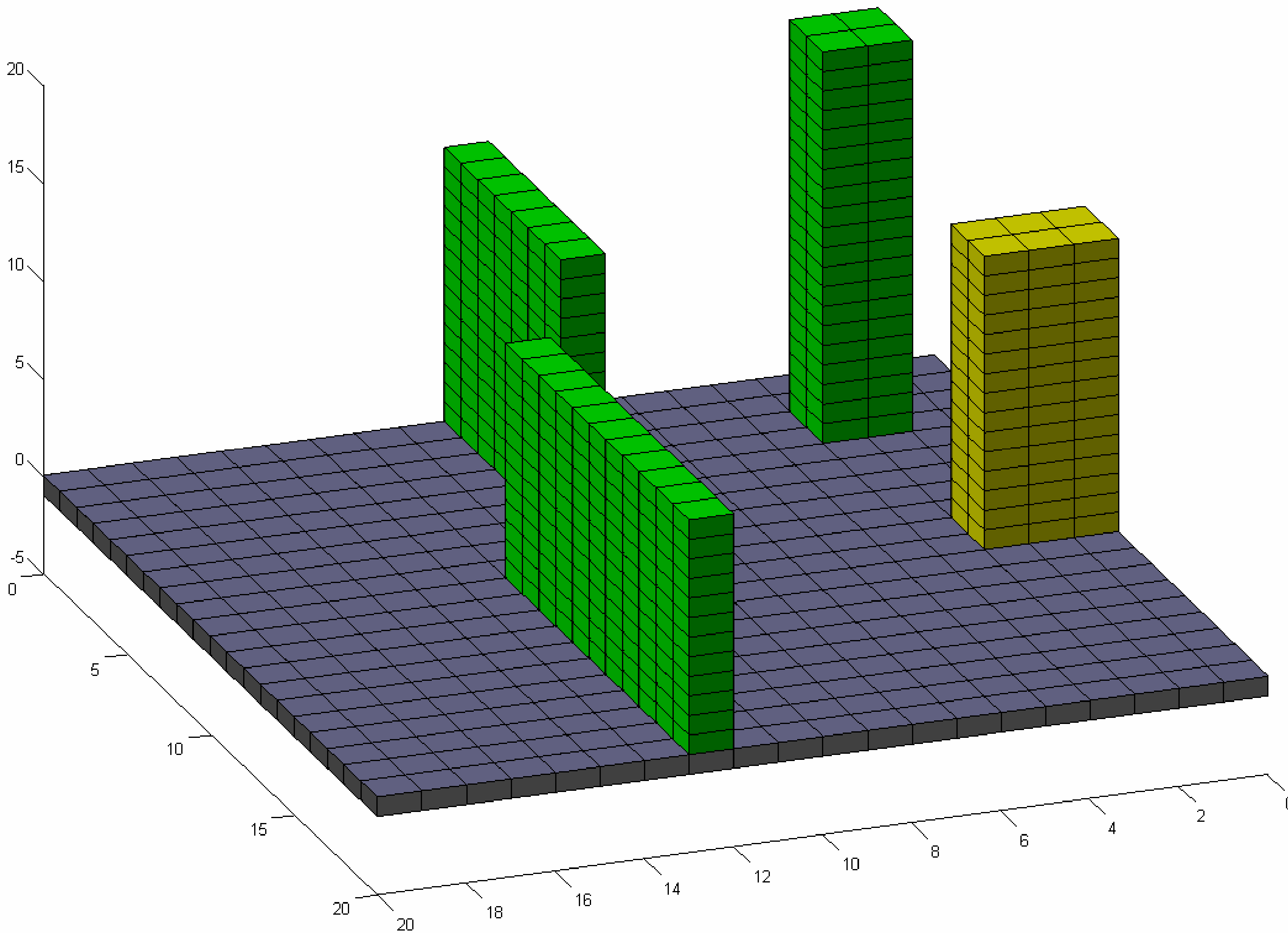
1074.
785.7
496.5
207.3
-81.8
-64.00
214.63
493.27
771.91
1050.54


1074.
785.7
496.5
207.3
-81.8
-64.00
214.63
493.27
771.91
1050.54

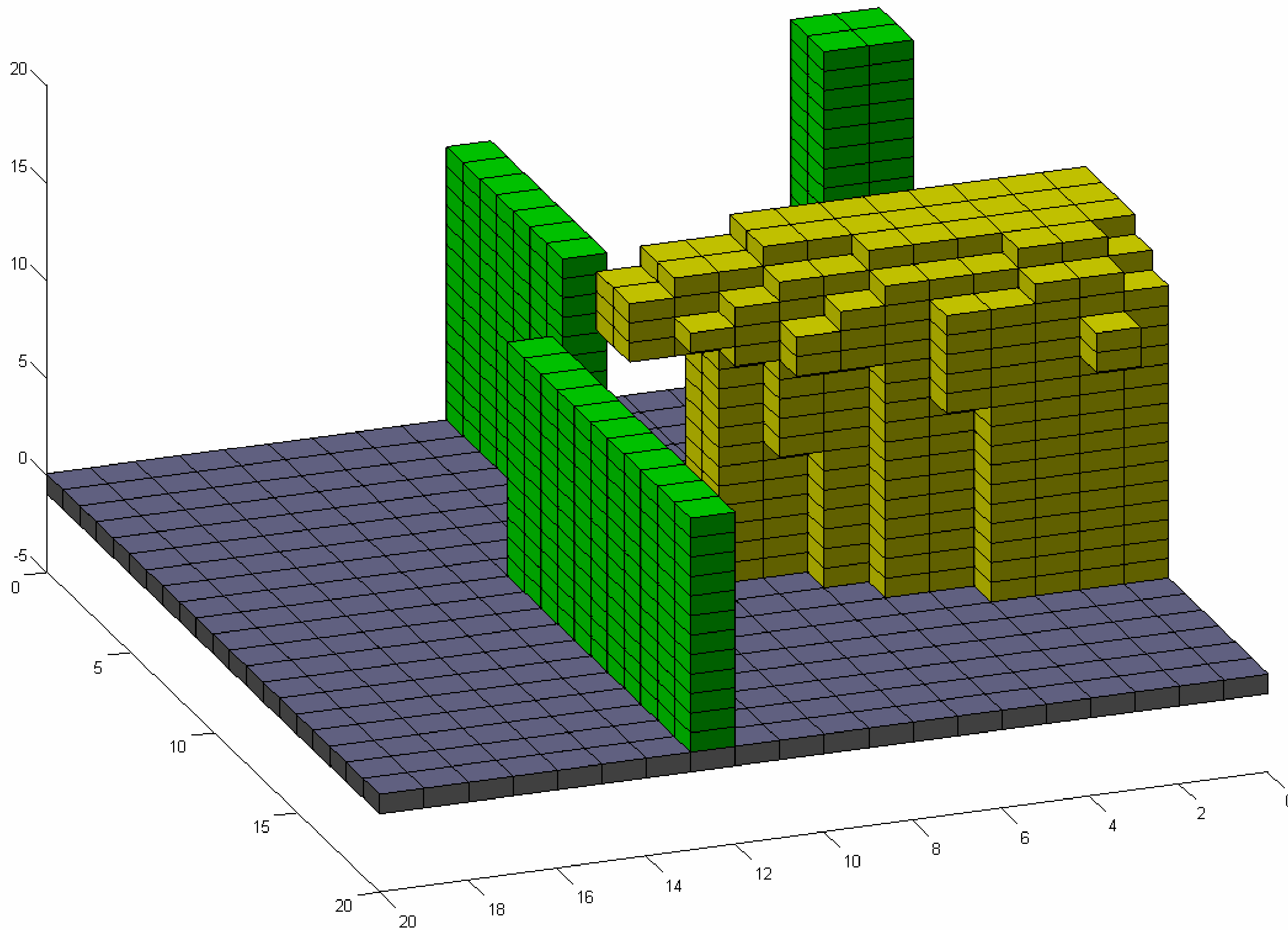

1074.
785.7
496.5
207.3
-81.8
-64.00
214.6?
493.27
771.91
1050.54



1074.
785.7
496.5
207.3
-81.8
-64.00
214.63
493.27
771.91
1050.54



20
15
10
5
0
-5
0
5
10
15
20
20
18
16
14
12
10
8
6
4
2
0



20
15
10
5
0
-5
0
5
10
15
20
20
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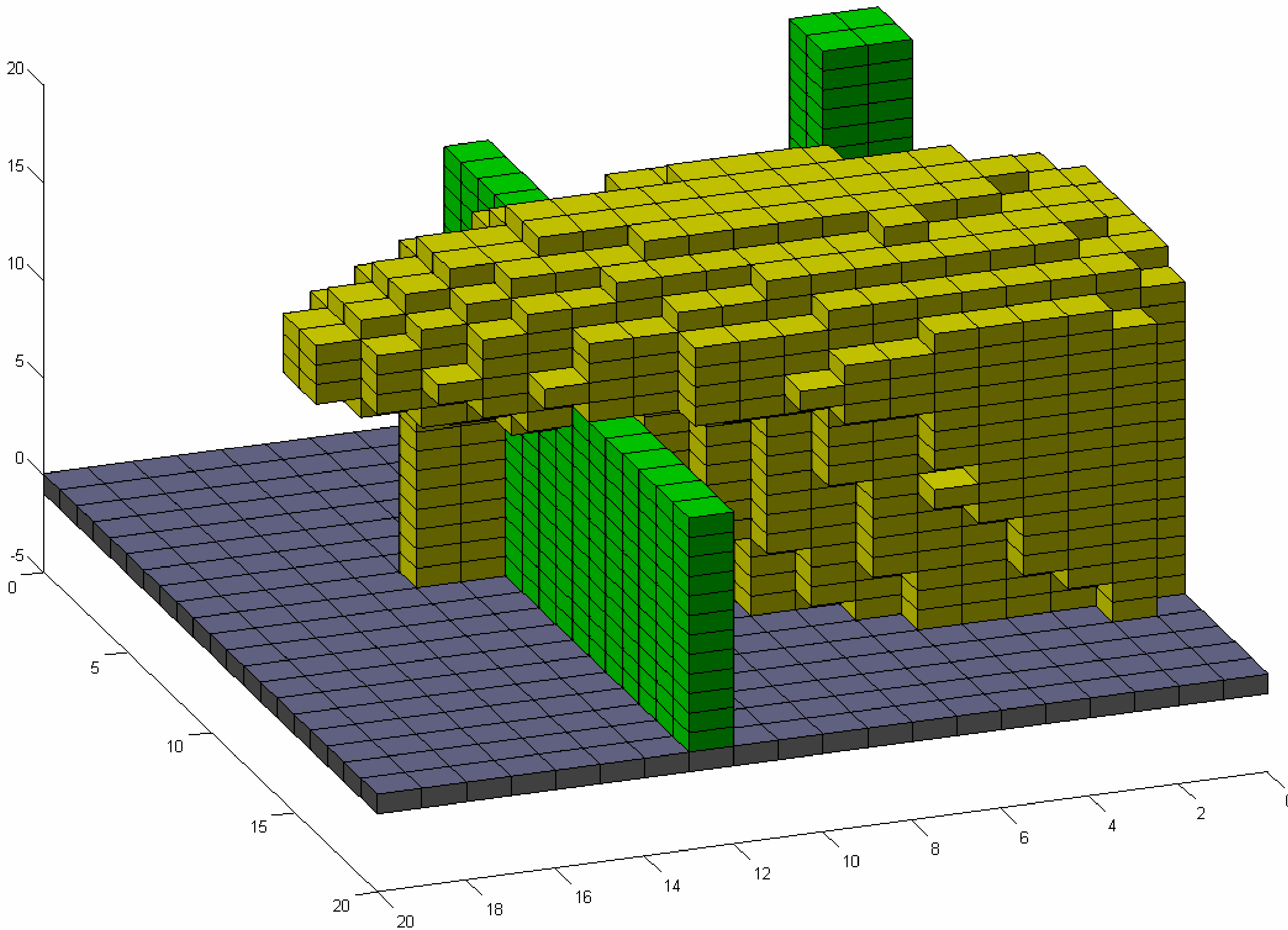
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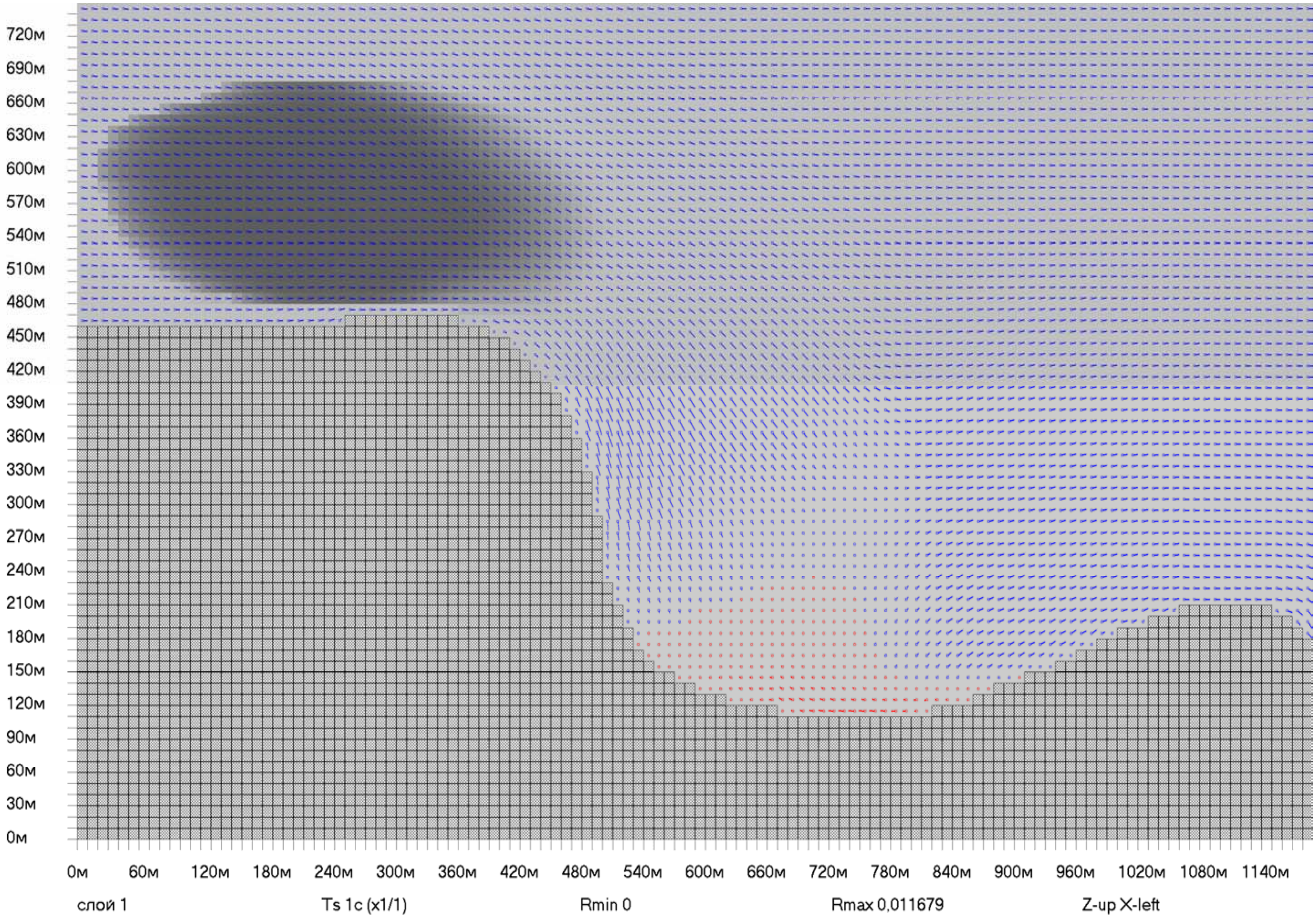
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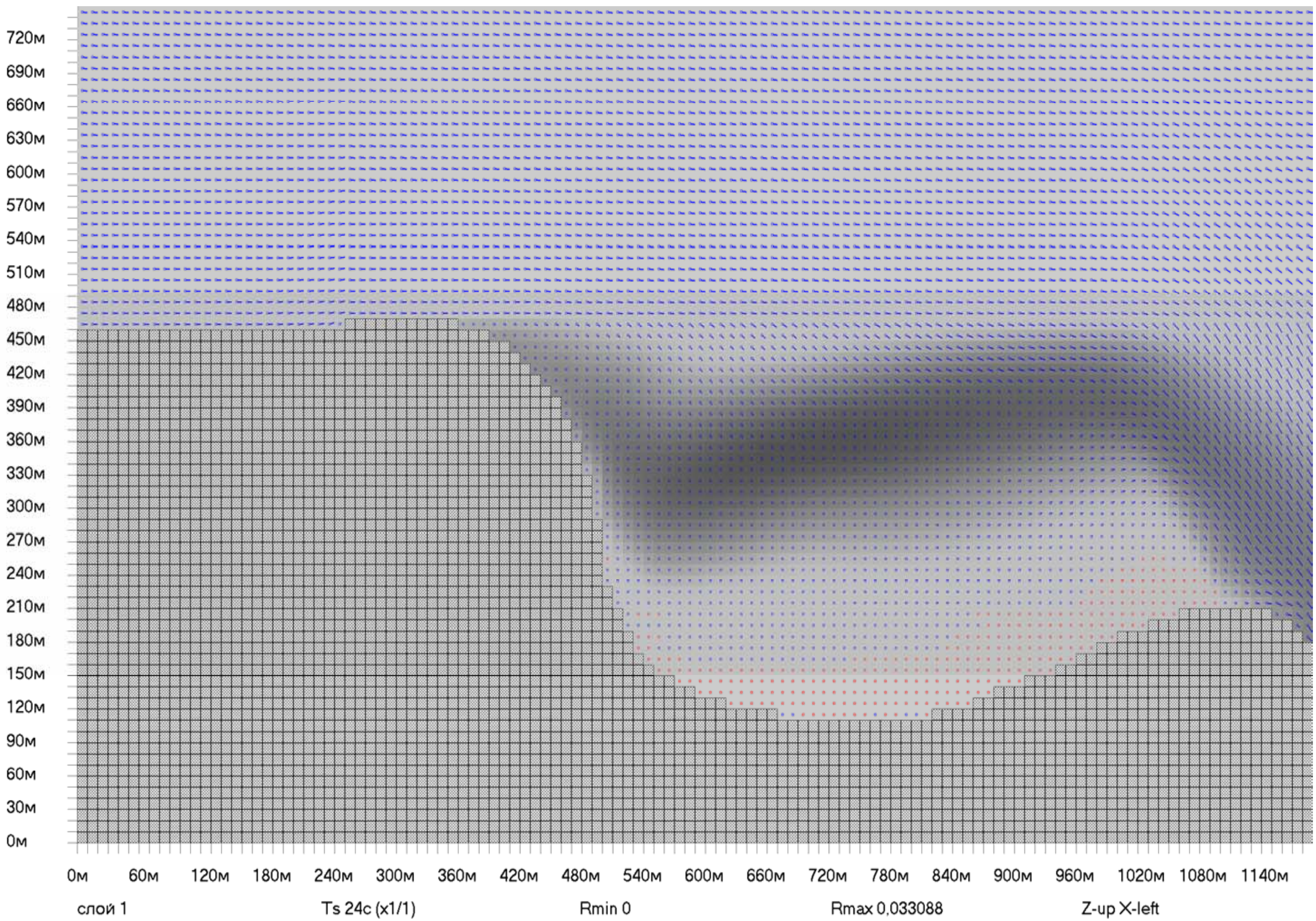
Z-up X-left



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слой 1
Ts 24c (x1/1)
Rmin 0
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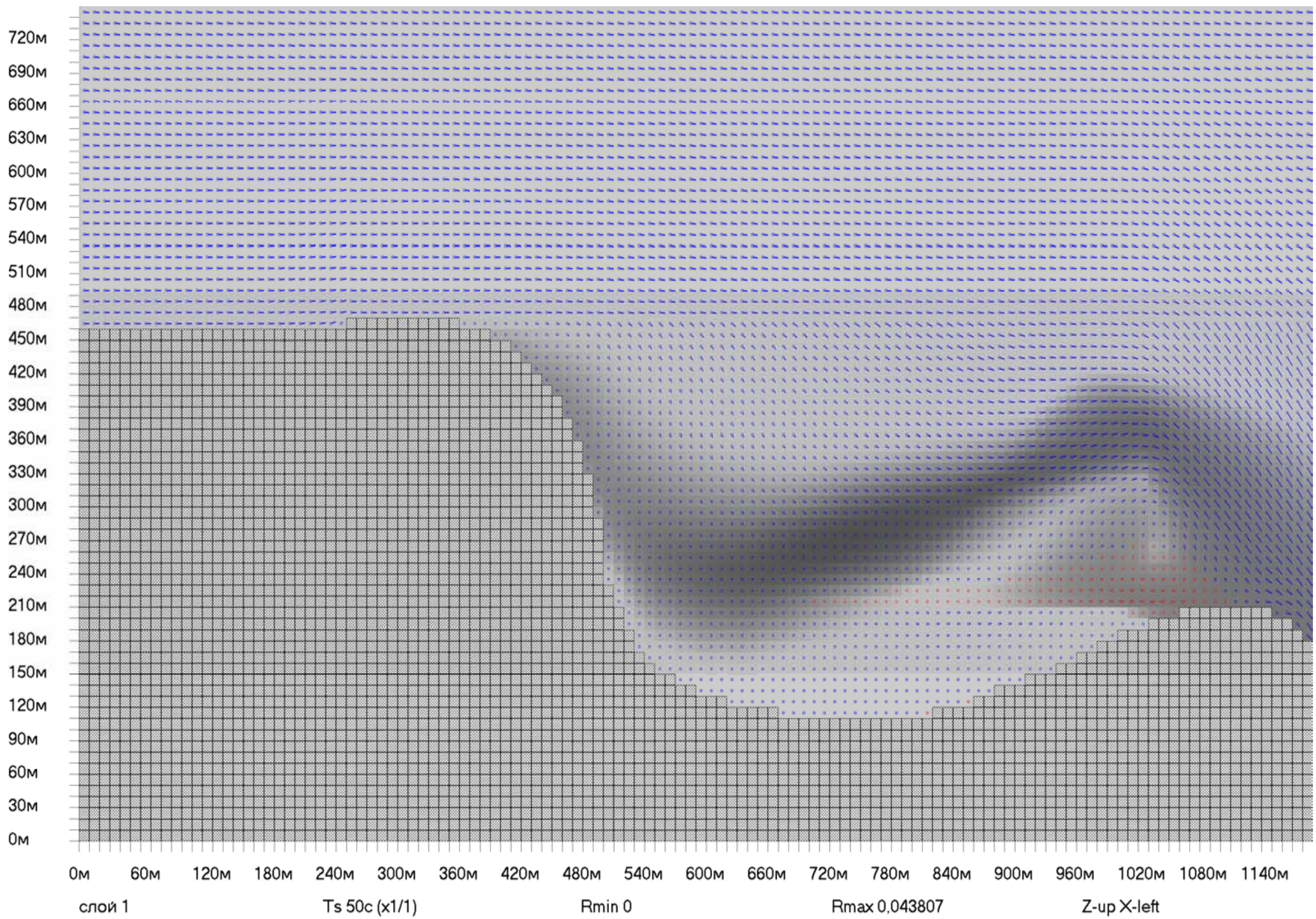
Z-up X-left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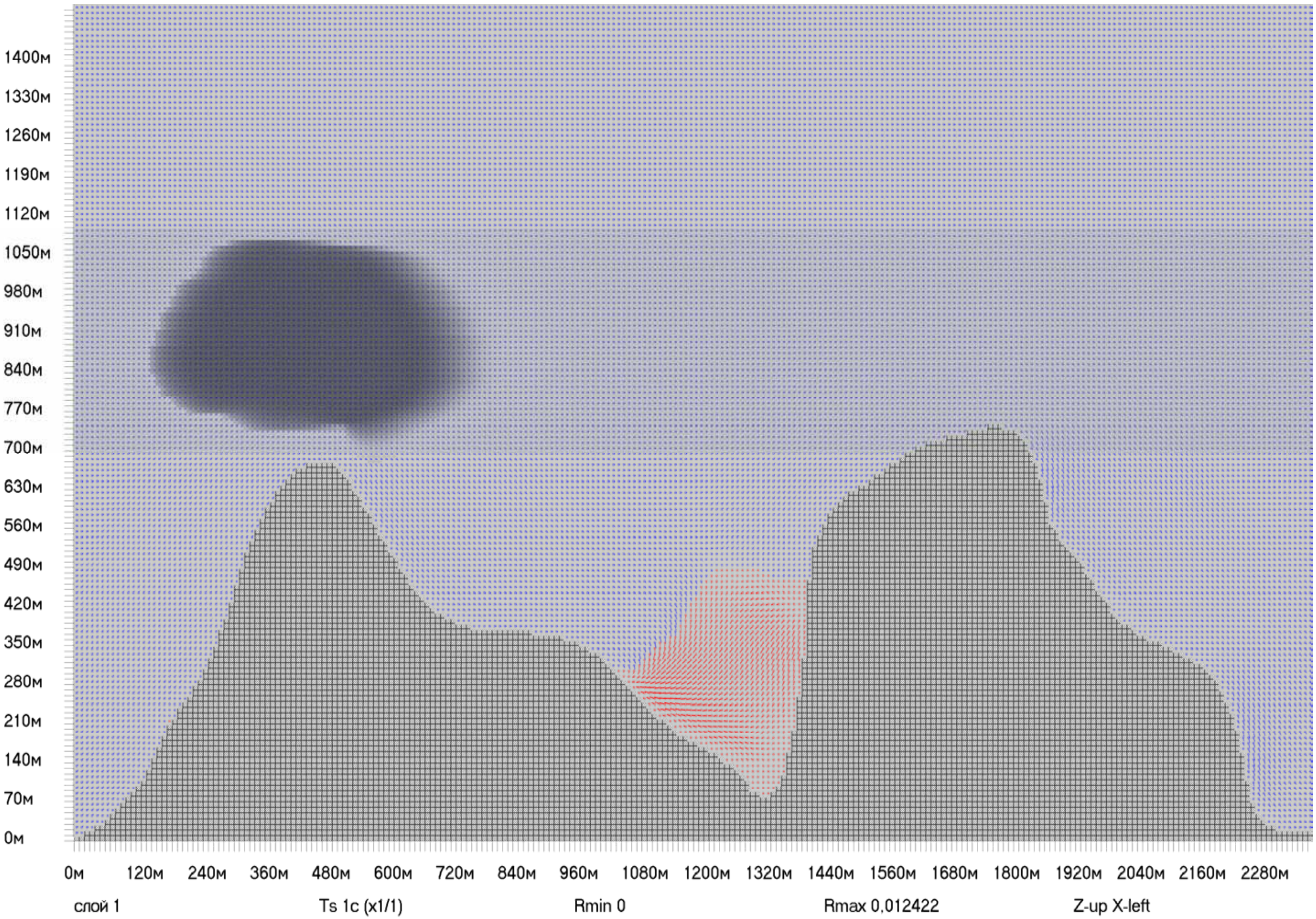
слой 1
Ts 50c (x1/1)
Rmin 0
Rmax 0,043807
Z-up X-left


1400м
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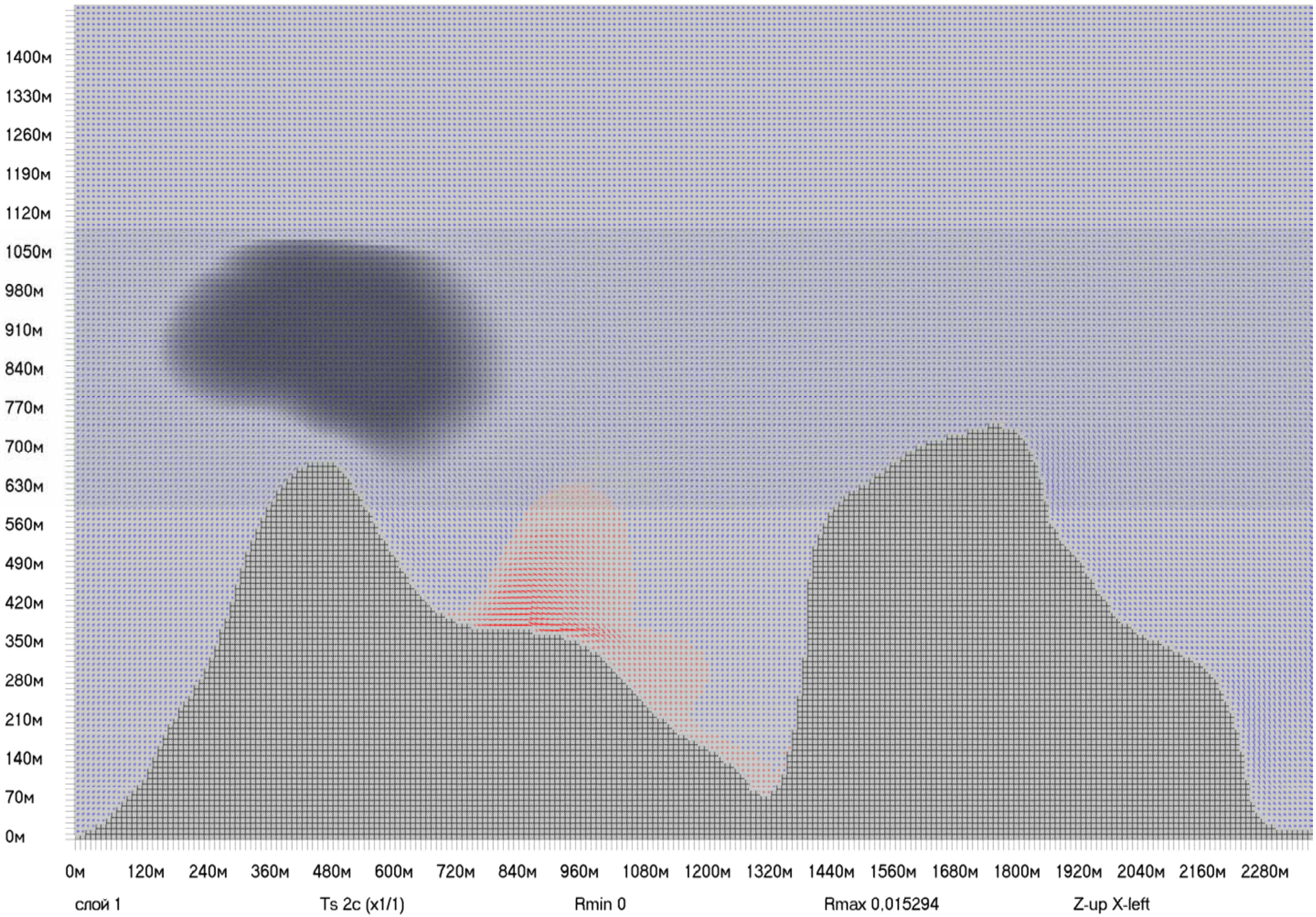
слой 1
Ts 1c (x1/1)
Rmin 0
Rmax 0,012422
Z-up X-left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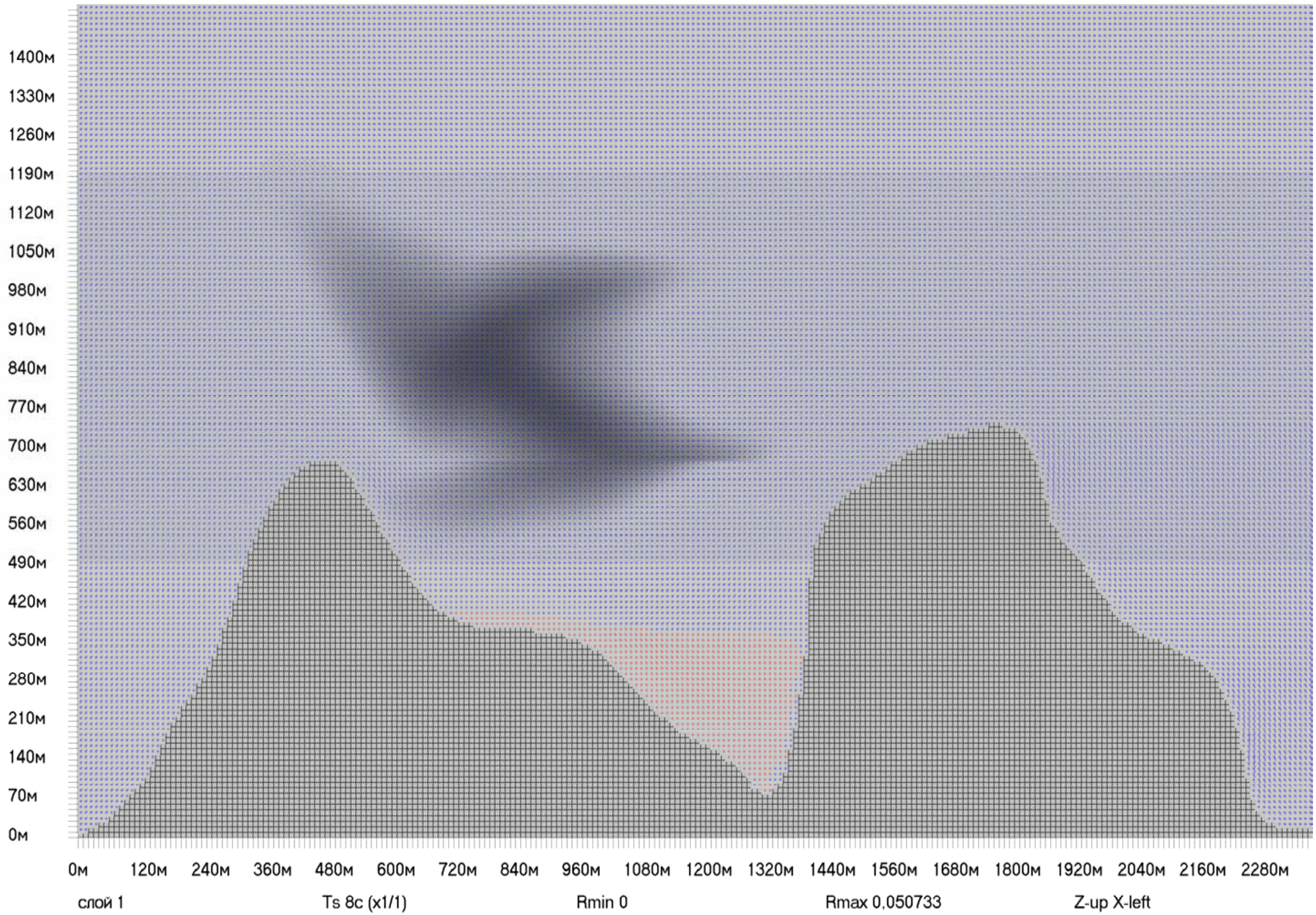
слой 1
Ts 2c (x1/1)
Rmin 0
Rmax 0,015294
Z-up X-left



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0м 120м 240м 360м 480м 600м 720м 840м 960м 1080м 1200м 1320м 1440м 1560м 1680м 1800м 1920м 2040м 2160м 2280м
слой 1
Ts 8c (x1/1)
Rmin 0
Rmax 0,050733
Z-up X-left