A quasi-normal scale elimination (QNSE) theory of stably stratified turbulence

Boris Galperin

University of South Florida, St. Petersburg, Florida

Semion Sukoriansky

Ben-Gurion University of the Negev, Beer-Sheva, Israel

Finnish Meteorological Institute
Helsinki, Finland
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Why QNSE?

- Stably stratified turbulence is one of the most difficult flow regimes for representation in numerical models of atmospheric boundary layers and numerical weather prediction
- “The most intrinsically difficult piece of a synthetic treatment is to account for nonlinearity and dissipation in the wave field” (Polzin, 2004)
- The QNSE model has been developed to systematically account for anisotropic turbulence and internal waves
- The QNSE model accurately predicts various flow characteristics unavailable in Reynolds stress models: turbulence spectra, flow anisotropization, dispersion relation of IW with turbulence, etc.
- The model is a viable alternative to Reynolds stress closures, is free of many of their limitations, and its implementation is straightforward

Basics of the QNSE theory of stably stratified turbulence

Consider a fully 3D tlow field with imposed vertical stabilizing temperature gradient. The flow is governed by

momentum

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} - \alpha g T \hat{e}_3 = \nu_0 \nabla^2 \mathbf{u} - \frac{\nabla P}{\rho} + \mathbf{f}_0$$

temperature

$$\frac{\partial T}{\partial t} + (\mathbf{u}\nabla)T + \frac{d\Theta}{dz}\mathrm{u}_3 = \kappa_0 \nabla^2 T$$

continuity

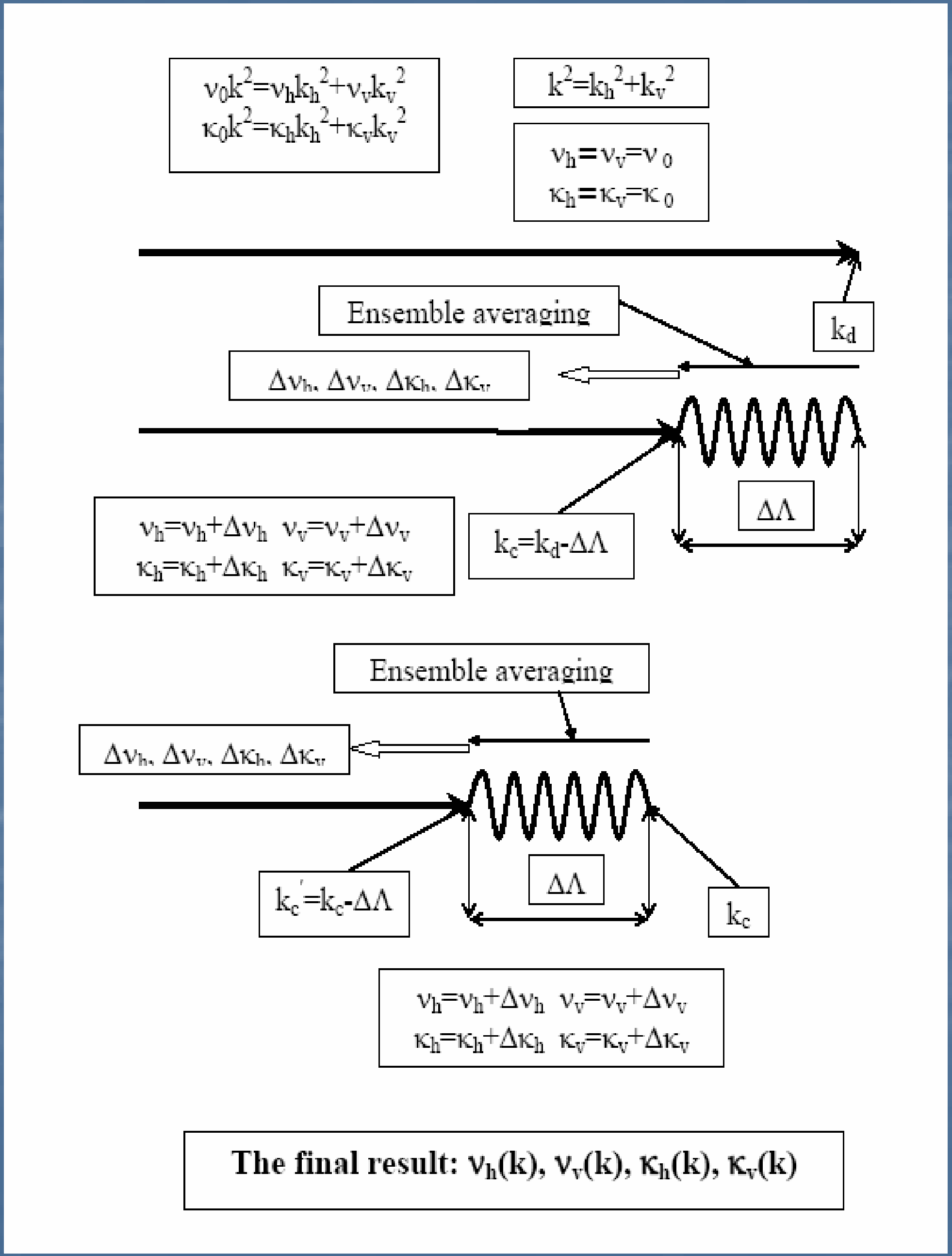
$$\nabla \mathbf{u} = 0$$

equations in Boussinesq approximation

The central problems: nonlinearity and coupling between u and T. They can be both addressed rigorously in the framework of a spectral theory.

Schematic representation of QNSE

General idea: Re is small for smallest scales of motion ➔



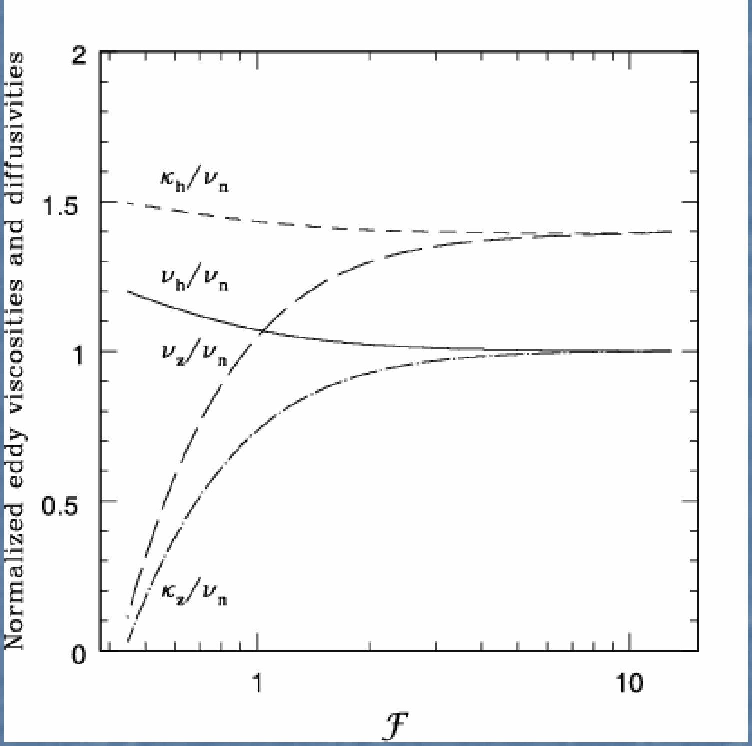
QNSE results

- Partial scale elimination yields a subgrid-scale model for LES
- Complete scale elimination yields eddy viscosities and eddy diffusivities for RANS (Reynolds-average Navier-Stokes) models

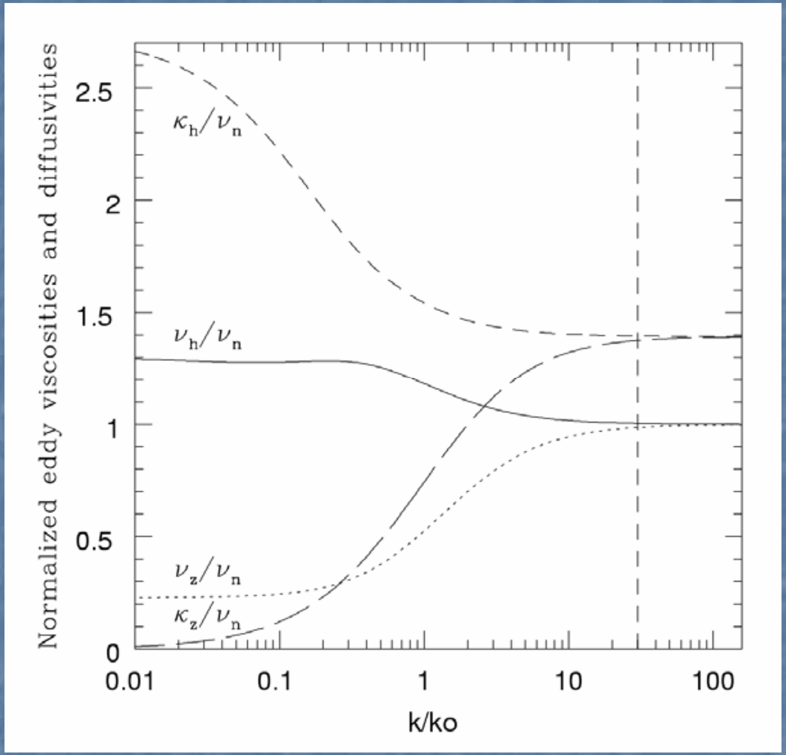
In either case, we don’t separate turbulence and waves and treat them as one entity

Analytical results

We obtain a coupled system of 4 differential equations for scale-dependent, horizontal and vertical eddy viscosities and eddy diffusivities. The system is solved analytically for weak and numerically for arbitrary stratification



Weak stratification



Arbitrary stratification

Dashed vertical line shows the threshold of internal wave generation in the presence of turbulence; $k_O = (N^3/\varepsilon)^{1/2}$ is the Ozmidov wave number

Weak stable stratification - Limitations of early theories

Lumley-Shur theory predicts an expression for a scale-dependent rate of the spectral energy transfer:

$$\overline{\epsilon}^{2/3} = \epsilon^{2/3}[1 + (k/k_O)^{-4/3}]$$

ε - rate of the viscous dissipation

“One of the weaknesses in this and other theories is that anisotropy is not adequately addressed” – Holloway (1988)

In the limit of weak stratification, the QNSE theory yields expansion in the powers of the spectral Froude number, $\mathcal{F} \simeq 0.5(k/k_O)^{2/3}$

$$\nu_h/\nu_n = 1 + 0.095\,\mathcal{F}^{-2},$$
$$\nu_z/\nu_n = 1 - 0.31\,\mathcal{F}^{-2},$$
$$\kappa_h/\nu_n = \alpha + 0.054\,\mathcal{F}^{-2},$$
$$\kappa_z/\nu_n = \alpha - 0.4\,\mathcal{F}^{-2},$$

$$\nu_n = 0.46\,\epsilon^{1/3}k^{-4/3}$$

$k_O = (N^3/\epsilon)^{1/2}$ is the Ozmidov wavenumber
$\alpha = Pr_t^{-1} = \nu_n/\kappa_n = 0.72$ is the inverse turbulent Prandtl number in neutral flows
ν_n and κ_n are eddy viscosity and eddy diffusivity in neutral flows

Replacing ε by $\bar{\varepsilon}$ in the QNSE expression for ν_n we find:

$$\nu = 0.46\epsilon^{1/3}k^{-4/3}[1 + (k/k_O)^{-4/3}]^{1/2} \simeq \nu_n[1 + 0.5(k/k_O)^{-4/3}]$$

or $$\nu/\nu_n = 1 + 0.5(k/k_O)^{-4/3}$$

This resembles the QNSE expression for the horizontal viscosity,

$$\nu_h/\nu_n = 1 + 0.4\ (k/k_O)^{-4/3}$$

This is confusing and intrinsically inconsistent because the Lumley-Shur theory accounts neither for turbulence anisotropy nor for the decrease of the vertical viscosity.

The Lumley-Shur theory predicts a spectral transition from the classical Kolmogorov $k^{-5/3}$ slope to the buoyancy stipulated k^{-3} slope. This transition, however, only takes place for the 1D, vertical spectrum and not for the 3D spectrum.

1D spectra can be calculated using QNSE

$$E_3(k_1) = \frac{8}{(2\pi)^4}\int U_{33}(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}k_2\,\mathrm{d}k_3 = 0.626\,\varepsilon^{2/3}k^{-5/3} - 0.704\,N^2 k_3^{-3}$$

$$E_3(k_3) = \frac{8}{(2\pi)^4}\int U_{33}(\omega,\boldsymbol{k})\,\mathrm{d}\omega\,\mathrm{d}k_1\,\mathrm{d}k_2 = 0.47\,\varepsilon^{2/3}k^{-5/3} - 0.143\,N^2 k_3^{-3}$$

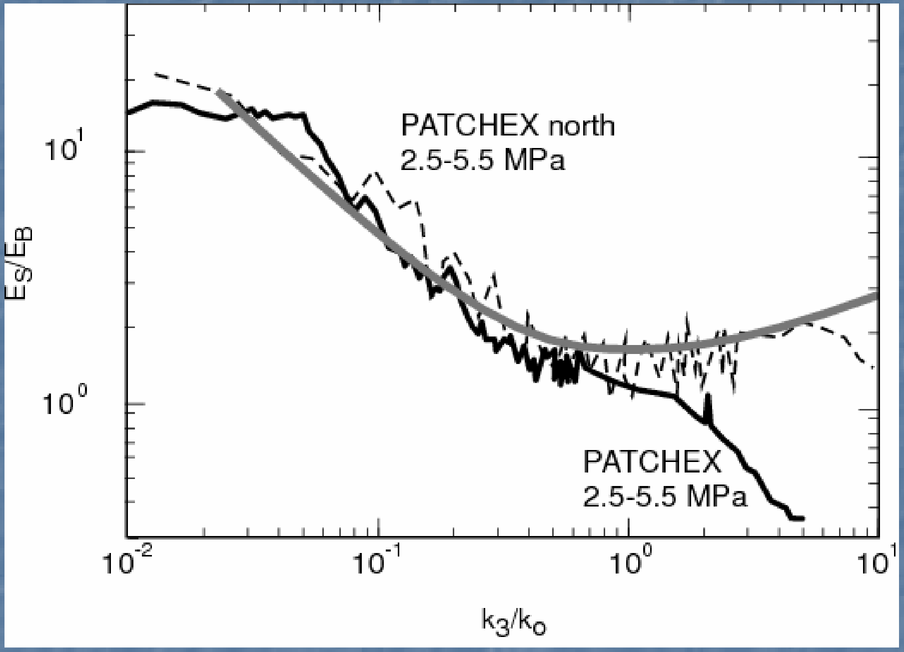
Vertical spectrum of horizontal velocity

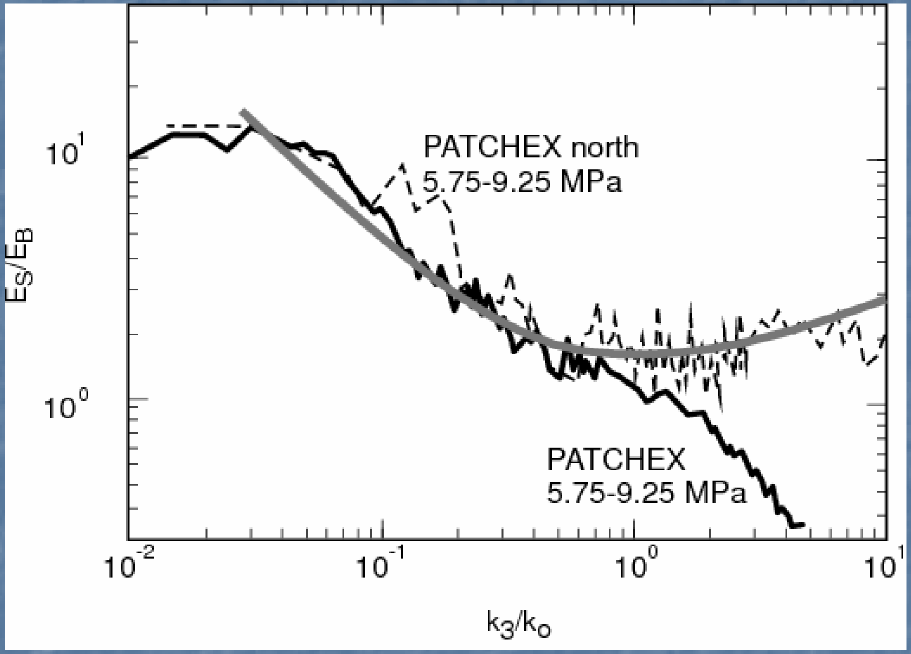
$$E_1(k_3) = \frac{8}{(2\pi)^4}\int U_{11}(\omega,\mathbf{k})d\omega dk_1 dk_2 = 0.626\epsilon^{2/3}k_3^{-5/3} + 0.214N^2k_3^{-3}$$

The anisotropization manifests itself as energy increase in the horizontal velocity components at the expense of their vertical counterpart

Comparison with observational data by Gargett et al., JPO, 1981: "A composite spectrum of vertical shear in the upper ocean"

$$\frac{E_S}{E_B} = \frac{2k_3^2 E_1(k_3)}{(\epsilon N)^{1/2}} = F(k_3/k_O) = 2 * 0.626(k_3/k_O)^{1/3}\left[1 + 0.34(k_3/k_O)^{-4/3}\right]$$





Gregg, Winkel, Sanford, JPO (1993)

"The results are dramatic, achieving a better collapse than obtained by Gargett et al. The collapse extends across the internal wave range ..."

Another example - atmospheric spectra

The QNSE theory predicts the vertical spectrum which is in a good agreement with the spectra observed in the stratosphere, troposphere, mesosphere, and thermosphere (our prediction is well approximated by the dashed line).

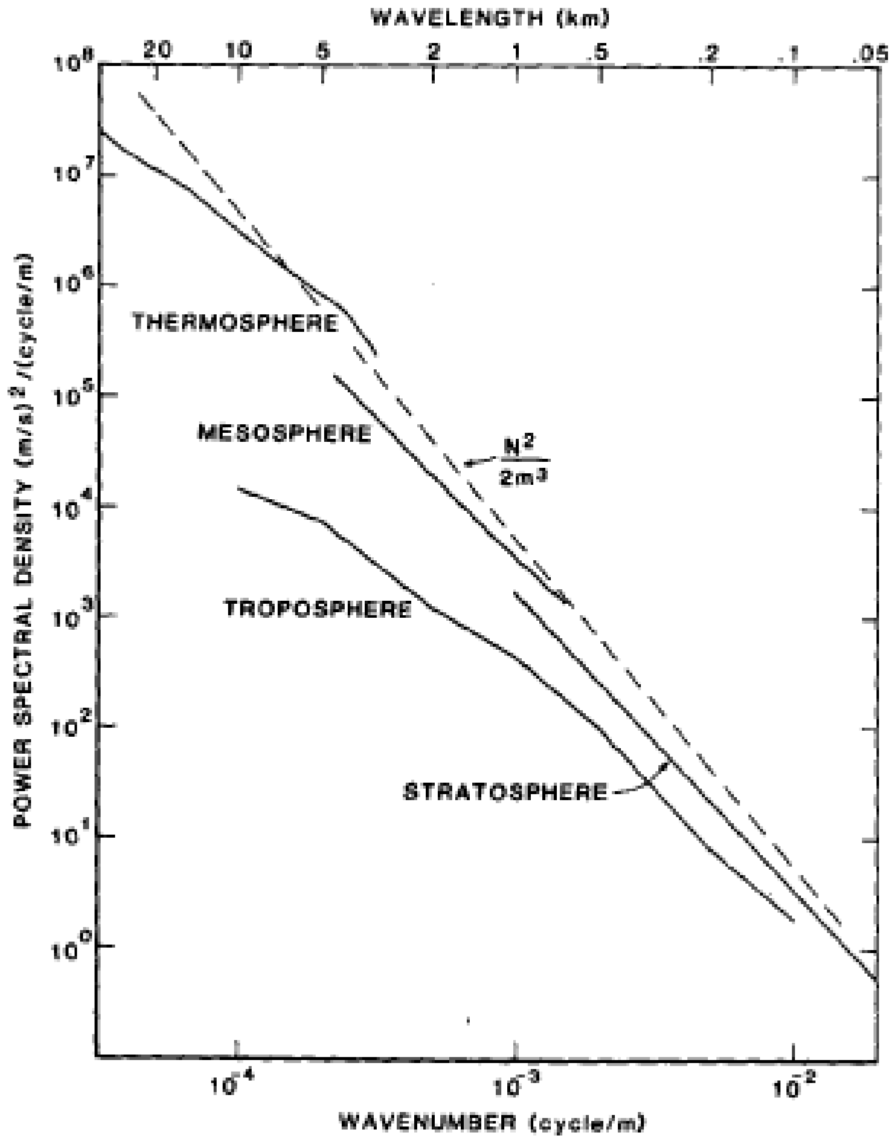


FIG. 1. Spectra of horizontal velocity versus vertical wavenumber as a function of altitude.

S. Smith et al., JAS, 1987

Internal waves and turbulence

The dispersion relation of internal waves with the effect of turbulence:

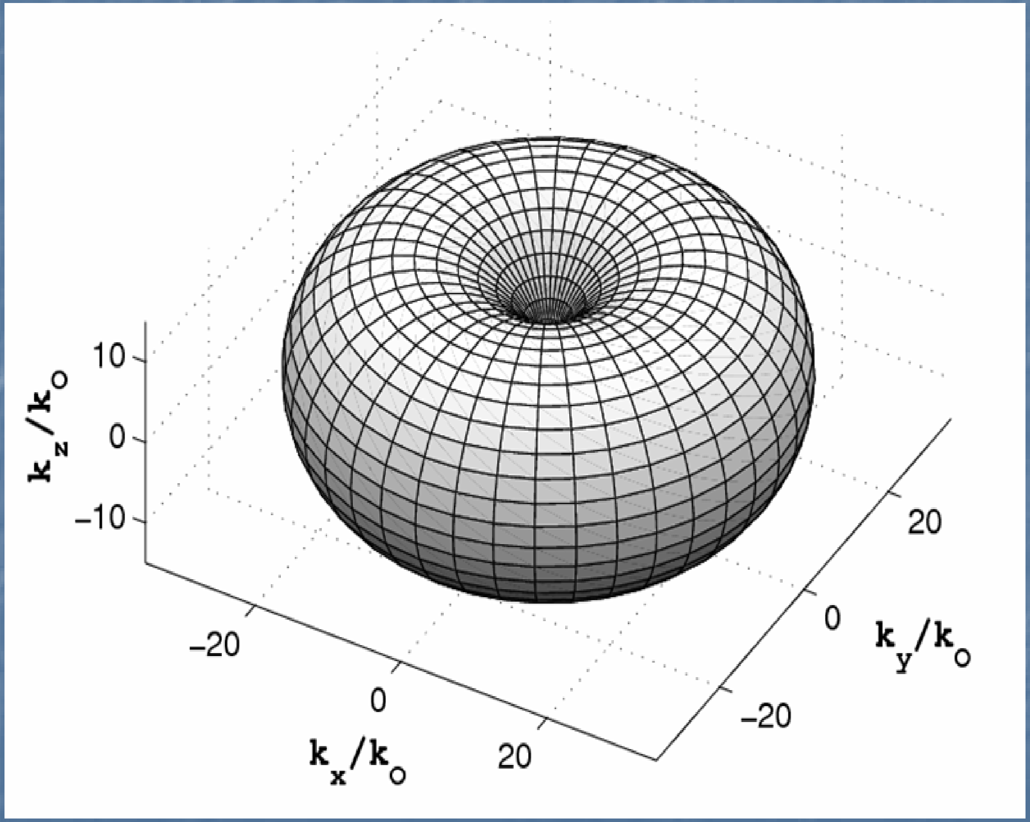
$$\omega^2 = N^2 \sin^2\theta \left\{ 1 - \left(\frac{k}{k_O}\right)^{4/3} \left[\frac{\left(\frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n}\right)\cos^2\theta + \left(\frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n}\right)\sin^2\theta}{4\sin\theta} \right]^2 \right\}$$

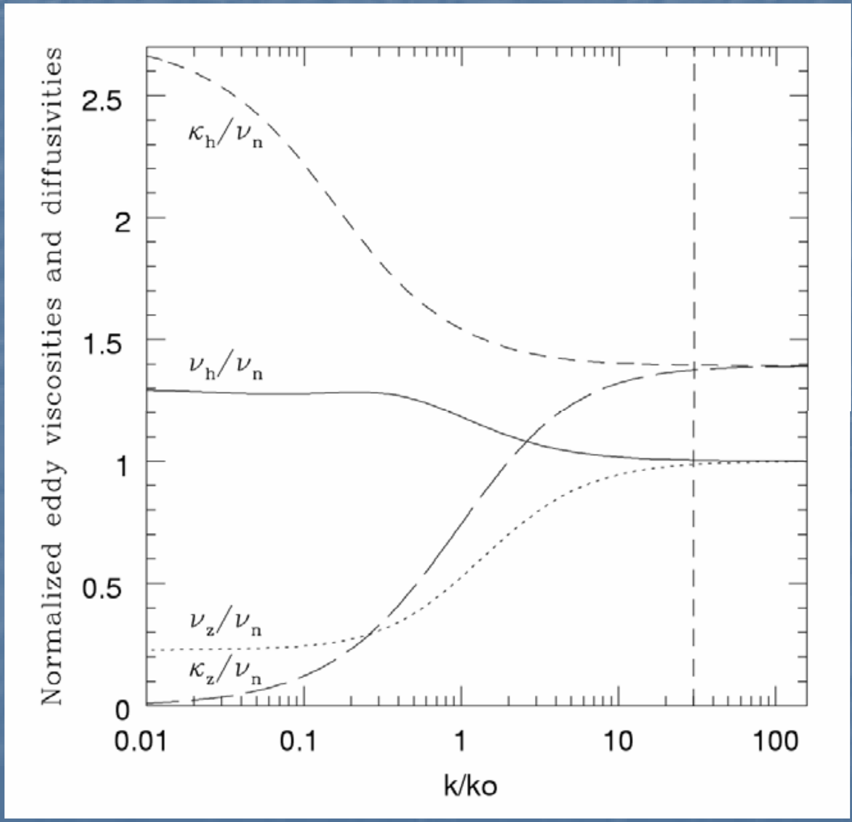
The limit of strong stratification ➔ classical dispersion relation for linear waves, **$\omega = N \sin\theta$.** Turbulence dominates at small scales. Criterion for wave generation is $\omega^2 \geq 0$ which gives

$$k_t = k_O \left| \frac{4\sin\theta}{\left(\frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n}\right)\cos^2\theta + \left(\frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n}\right)\sin^2\theta} \right|^{3/2} \simeq 32 k_O |\sin\theta|^{3/2}$$

This equation describes a torus

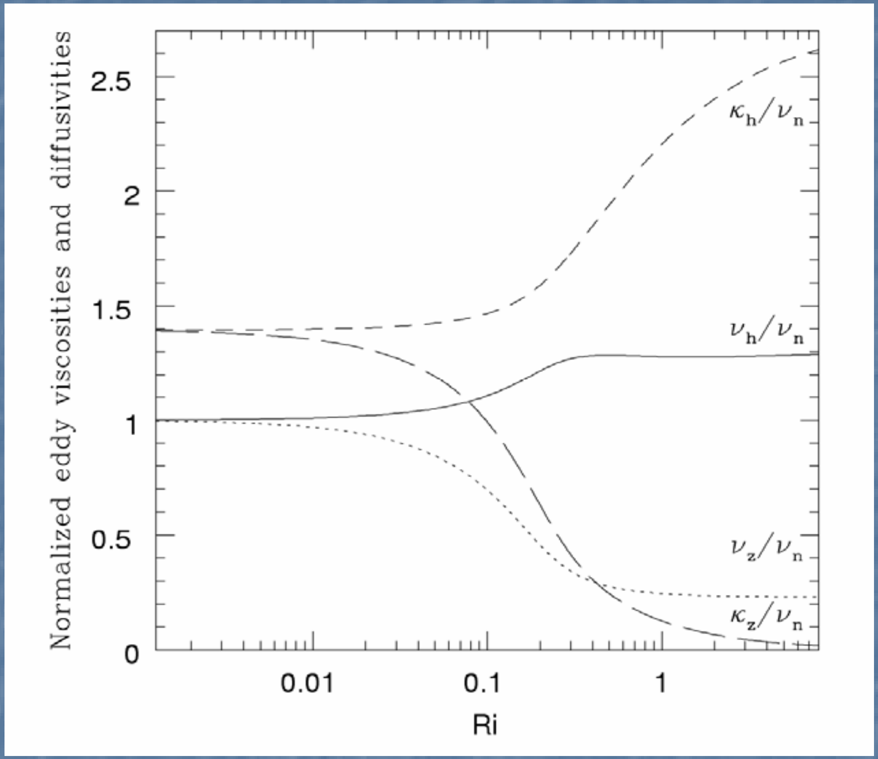
Internal waves exist only for the wavenumbers embedded inside the torus





RANS modeling

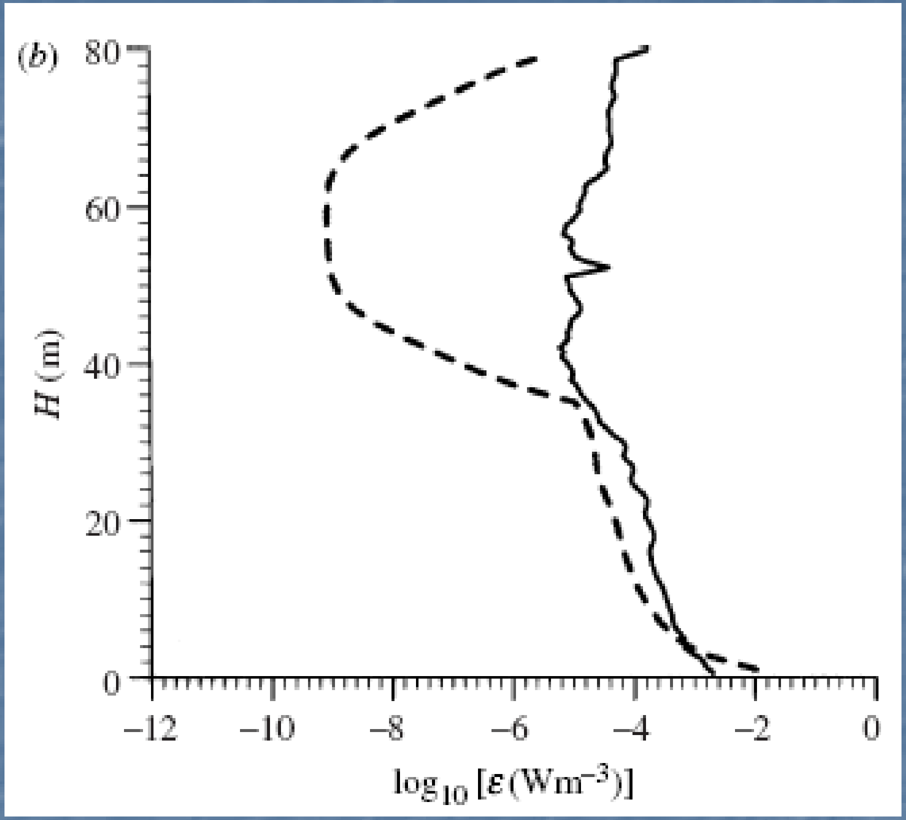
Eliminate all fluctuating scales; recast turbulent exchange coefficients as functions of **the gradient Richardson number, Ri = N^2 / S^2**

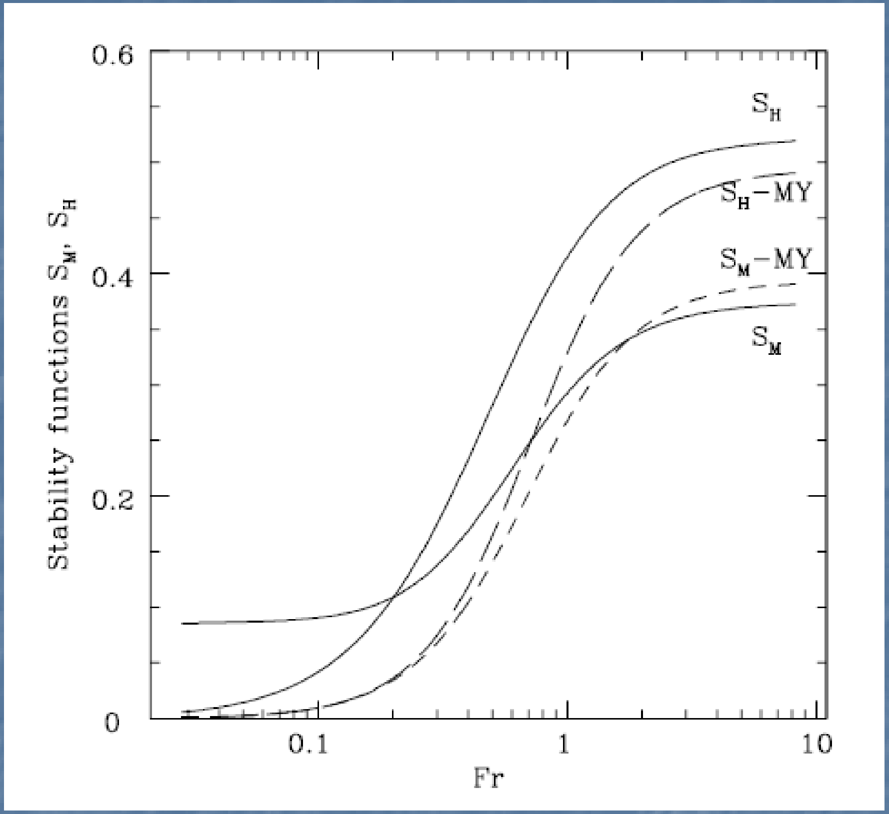


ν_z and κ_z decrease with Ri while ν_h and κ_h increase with Ri

Momentum mixing by internal waves ➔
ν_z decreases slower than κ_z
ν_z remains finite even at very large Ri

Stability functions S_M and S_H from the QNSE model and from the Mellor-Yamada model modified by Galperin et al. (1988)

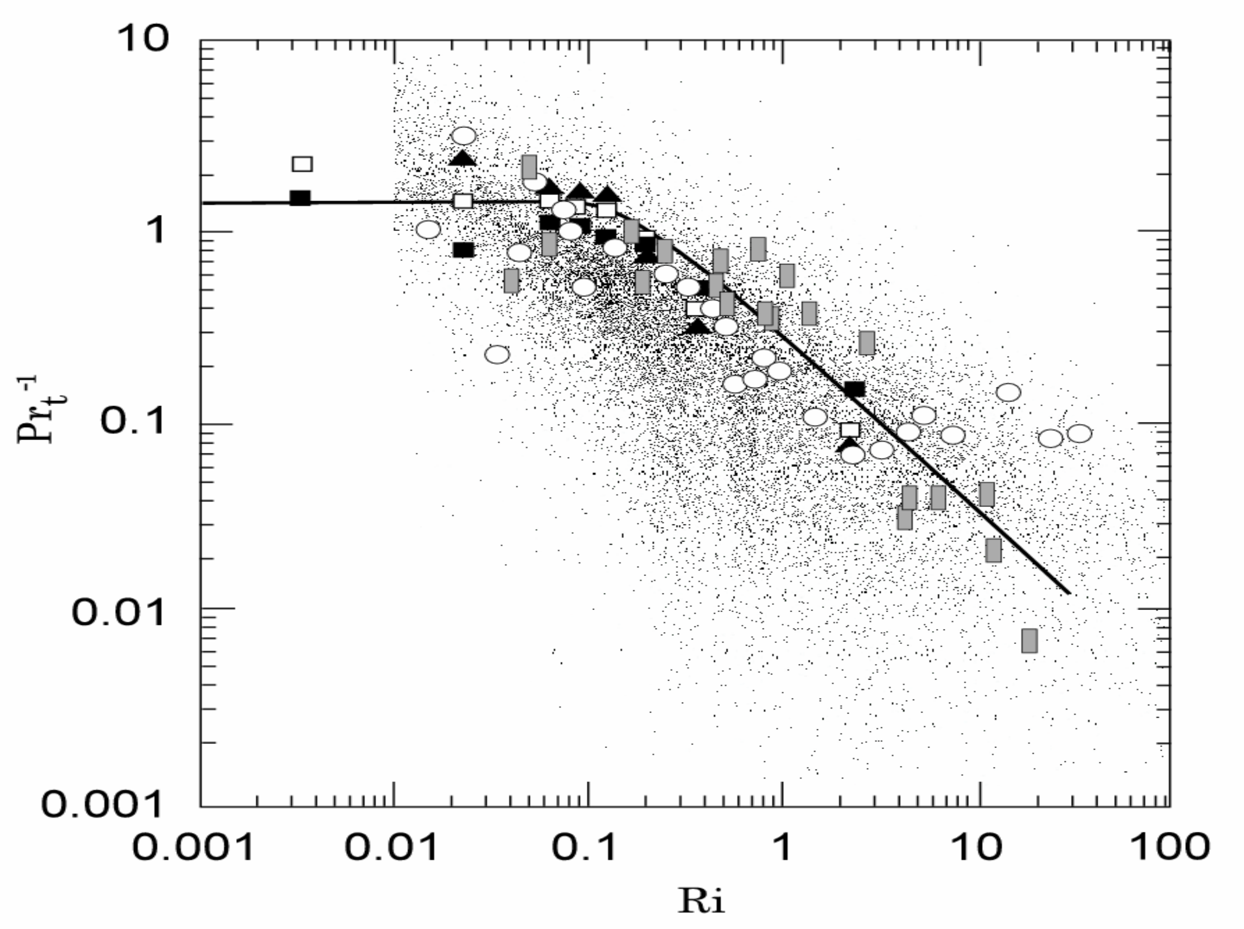




Temporal mean profile of ε in the continental shelf seas. Solid line – data; dashed line - an estimate by M-Y level 2 scheme (Rippeth, 2005; Simpson et al. 1996). The M-Y model underpredicts the mixing.

QNSE model is not expected to underpredict mixing in strong stable stratification

Comparison with data: Pr_t as a function of Ri



Observational data by Monti et al. (2002) and from Halley Base, Antarctica, collected by the British Antarctic Survey

Both the data and the theory point to the absence of critical Ri and the dependence $Pr_t \propto Ri$ for large Ri

$Pr_t \propto Ri$ and $\nu_z S^2 - \kappa_z N^2 = \varepsilon$ yield $\kappa_z = c\, \varepsilon N^{-2}$ with c = 0.25 to 0.5 (in the range by Gregg, Osborn, Weinstock and others).

The linear dependency, $Pr_t \propto Ri$, for Ri > 0.2, has been used in some ocean circulation models to facilitate realistic results (Blanke and Delecluse, 1993; Meier, 2001).

All atmospheric and oceanic circulation models employ 'background' mixing coefficients in strong stable stratification

Zhang and Steele, JGR, 2007, Effect of vertical mixing on the Atlantic water layer circulation in the Arctic Ocean:

"Note that a number of vertical mixing schemes have been implemented in general ocean circulation models, including the traditional constant viscosity/diffusivity approach, the Richardson number-dependent scheme [Pacanowski and Philander, 1981], and the Mellor and Yamada [1982] turbulence closure scheme... Mixing below the surface mixed layer is strongly influenced by a ''background'' diffusivity... **Background viscosity is always ten times the background diffusivity, in keeping with the procedure at LANL.**"

Critical Richardson number

- Critical Richardson number is a value of Ri at which turbulent mixing ceases, flow becomes re-laminarized and can be described by isotropic molecular viscosity
- Atmospheric and oceanic data show that this rarely happens
- Furthermore, $Pr_t = f(Ri)$ is inconsistent with Ri_{cr}
- Yet Reynolds stress models often imply the existence of the critical Ri and underpredict the mixing
- QNSE theory is free of the critical Richardson number because it includes mixing by internal waves and flow anisotropization
- Latest Reynolds stress models (Zilitinkevich et al. (2007), Canuto et al. (2008)) are also designed without Ri_{cr}

Monin-Obukhov similarity theory

Monin-Obukhov length scale

$$L = \frac{u_*^2}{\kappa\beta\theta_*} = \frac{z\,\alpha_M^2(Ri)}{Ri\,\alpha_H(Ri)}$$

ζ = z/L can be presented as a function of Ri and vise versa.
Velocity and temperature at the lower grid point are

$$u_1 = \frac{u_*}{\kappa}\int_{z_0}^{z_1}\frac{dz}{z\alpha_M}$$

$$\theta_1 = \theta_0 + \frac{\theta_*}{\kappa}\int_{z_0}^{z_1}\frac{dz}{z\alpha_H}$$

The integrals are computed analytically as functions of z/L:

$$u_1 = \frac{u_*}{\kappa}\left(\ln\frac{z_1}{z_0} + c_u\frac{z_1}{L} + \ldots\right), \qquad c_u = 2.25$$

$$\theta_1 = \theta_0 + \frac{\theta_*}{1.4\kappa}\left(\ln\frac{z_1}{z_0} + c_\theta\frac{z_1}{L} + \ldots\right), \qquad c_\theta = 2.04$$

Validation of the QNSE-based RANS models in a single-column formulation in models of atmospheric boundary layers

K-ε format

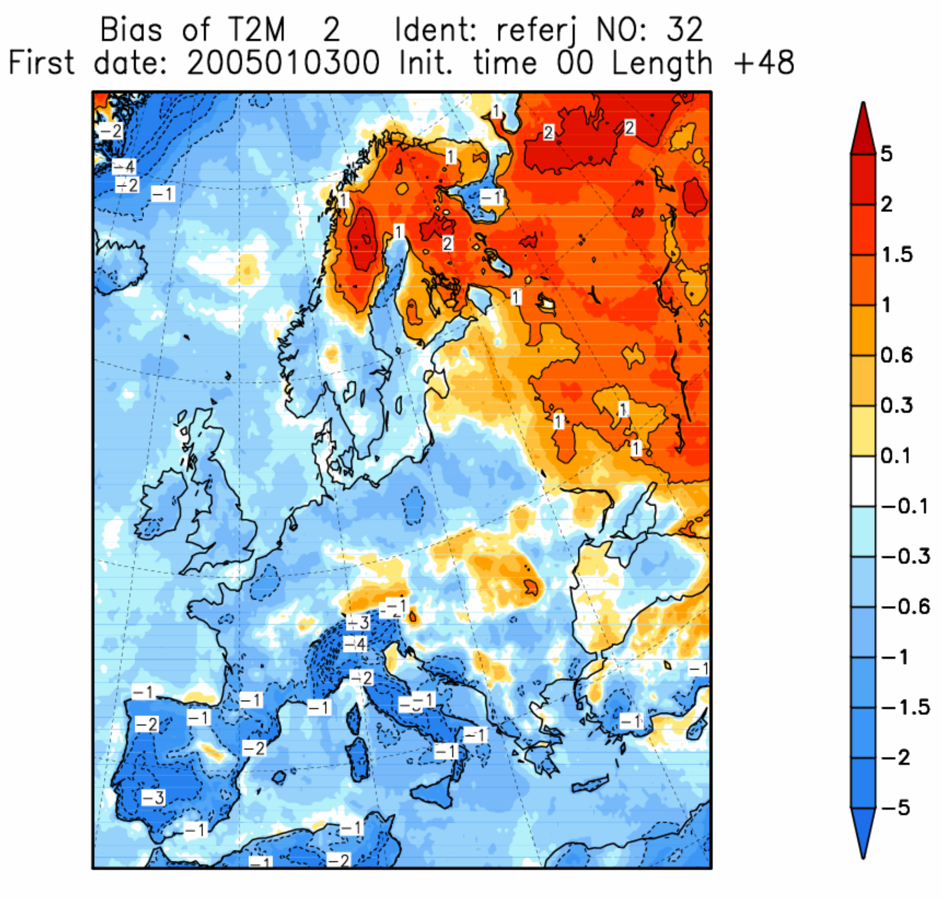
- BASE (GABLS 1)
- SHEBA
- CASES-99 (particularly, IOP-9 event)
- CASES-99 (GABLS 2)

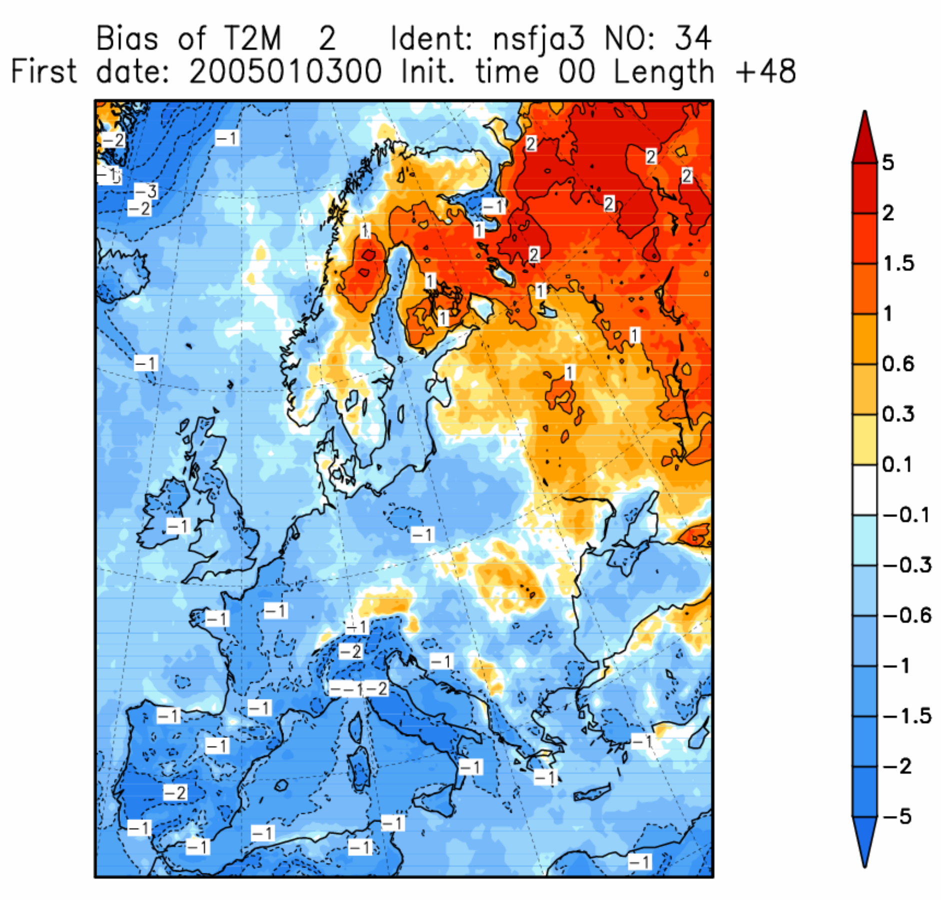
K-ℓ format

- HIRLAM 7.0 – CASES-99 (GABLS 2)
- January and March, 2005 – fully 3D
- WRF – work in progress

Implementation in HIRLAM

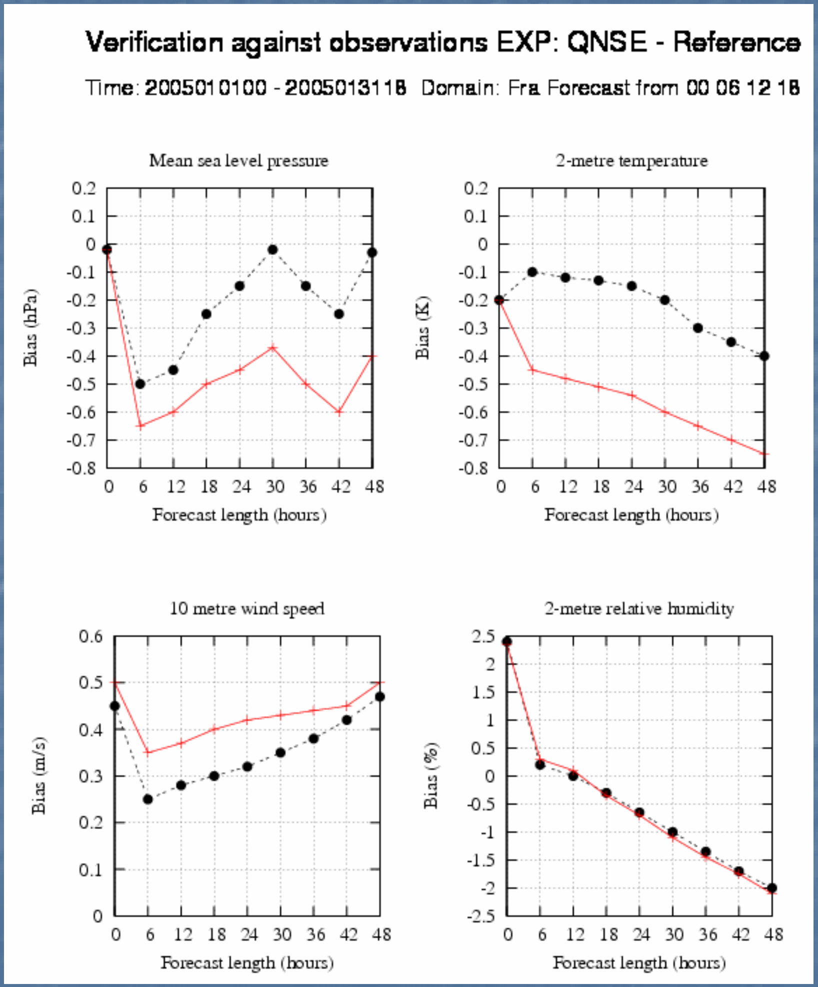
Bias of 2m temperature in simulations of January 2005



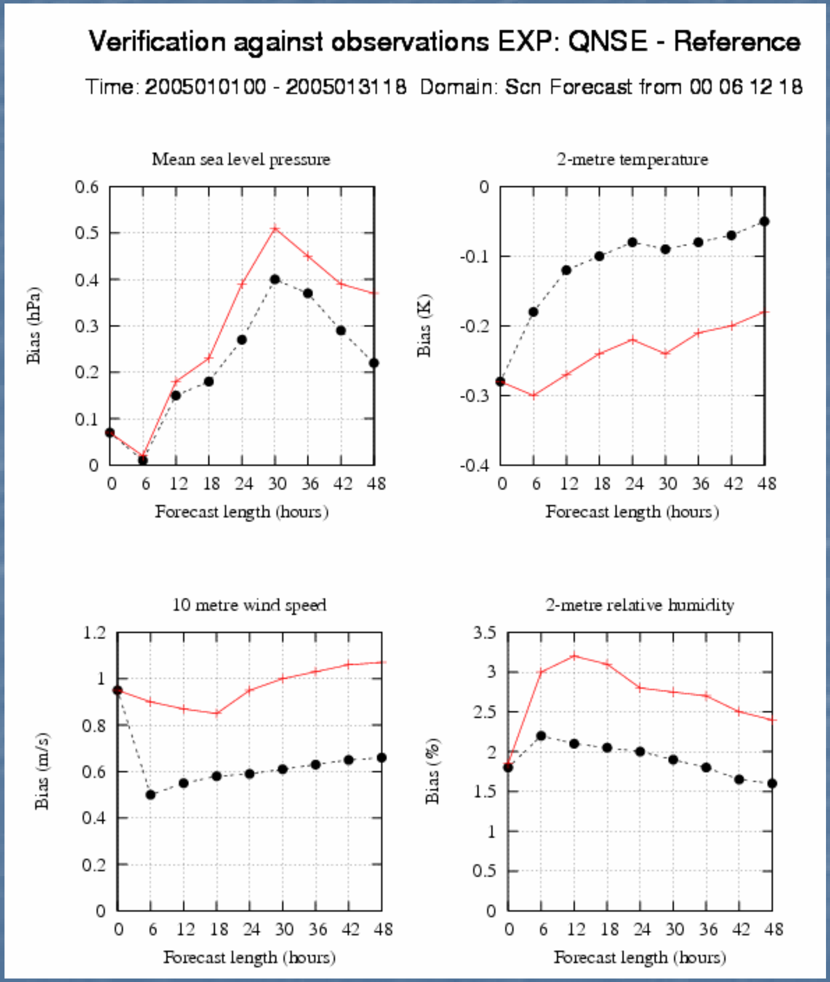


Bias of 2m temperature for January, 2005

Red lines – reference HIRLAM, black dots – our model

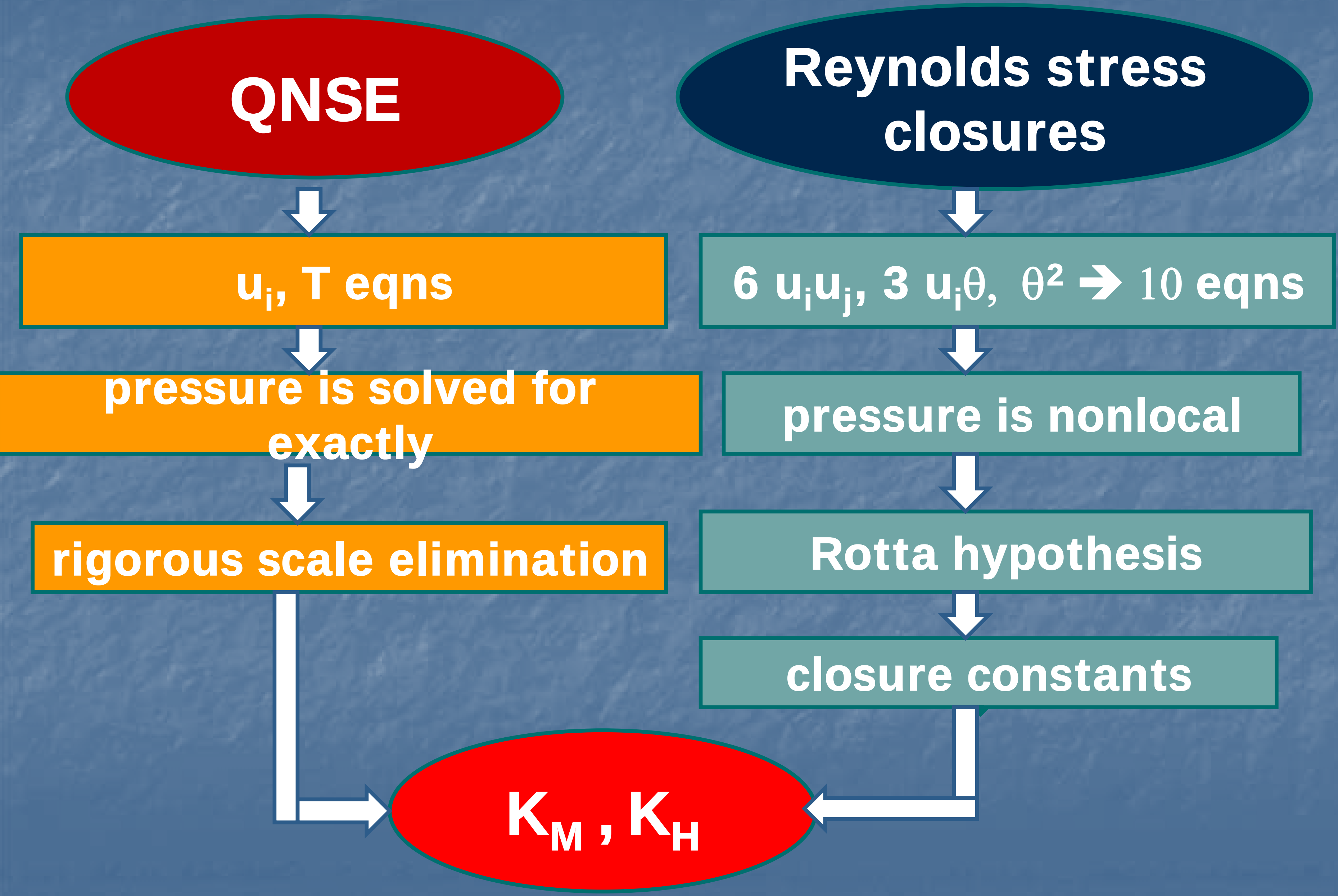


Skills of the forecast for France



Skills of the forecast for Scandinavia

QNSE vs. Reynolds stress closures



Conclusions

- Derivation of the QNSE model of turbulence is maximally proximate to first principles; resolves horizontal-vertical anisotropy and accounts for the combined effect of turbulence and internal waves
- QNSE clarifies early theories of stably stratified turbulence and emphasizes the importance of anisotropy
- Predicts correct behavior of Pr_t and is free of the critical Ri
- QNSE-based RANS models have been tested in both K-ε and K-ℓ formats
- QNSE-based K-ℓ model significantly improves turbulence representation in both HIRLAM and WRF
- QNSE model improves predictive skills of HIRLAM in +24h and +48h weather forecasts and is anticipated to have similar impact in WRF
- **QNSE-based models are a viable alternative to the Reynolds stress models**

Thank You!

Acknowledgments

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