A quasi-normal scale elimination (QNSE) theory of stably stratified turbulence

Boris Galperin
University of South Florida, St. Petersburg, Florida

Semion Sukoriansky
Ben-Gurion University of the Negev, Beer-Sheva, Israel

Finnish Meteorological Institute
Helsinki, Finland
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Stably stratified turbulence is one of the most difficult flow regimes for representation in numerical models of atmospheric boundary layers and numerical weather prediction.

“The most intrinsically difficult piece of a synthetic treatment is to account for nonlinearity and dissipation in the wave field” (Polzin, 2004)

The QNSE model has been developed to systematically account for anisotropic turbulence and internal waves.

The QNSE model accurately predicts various flow characteristics unavailable in Reynolds stress models: turbulence spectra, flow anisotropization, dispersion relation of IW with turbulence, etc.

The model is a viable alternative to Reynolds stress closures, is free of many of their limitations, and its implementation is straightforward.
Basics of the QNSE theory of stably stratified turbulence

Consider a fully 3D flow field with imposed vertical stabilizing temperature gradient. The flow is governed by

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} - \alpha g T \hat{e}_3 = \nu_0 \nabla^2 \mathbf{u} - \frac{\nabla P}{\rho} + \mathbf{f}_0
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{u} \nabla) T + \frac{d\Theta}{dz} u_3 = \kappa_0 \nabla^2 T
\]

\[\nabla \mathbf{u} = 0\]

Equations in Boussinesq approximation

The central problems: nonlinearity and coupling between \( u \) and \( T \). They can be both addressed rigorously in the framework of a spectral theory.
Schematic representation of QNSE

General idea: Re is small for smallest scales of motion

\[ \nu k^2 = \nu_h k_h^2 + \nu_v k_v^2 \]

\[ k^2 = k_h^2 + k_v^2 \]

\[ \nu_h = \nu_v = \nu_0 \]

\[ \kappa_h = \kappa_v = \kappa_0 \]

Ensemble averaging

\[ \Delta \nu_h, \Delta \nu_v, \Delta \kappa_h, \Delta \kappa_v \]

\[ \nu_h = \nu_h + \Delta \nu_h \]

\[ \nu_v = \nu_v + \Delta \nu_v \]

\[ \kappa_h = \kappa_h + \Delta \kappa_h \]

\[ \kappa_v = \kappa_v + \Delta \kappa_v \]

\[ k = k - \Delta \Lambda \]

\[ k = k_d + \Delta \Lambda \]

\[ \Delta \Lambda \]

\[ \kappa_c = k_c - \Delta \Lambda \]

\[ k_c \]

The final result: \( \nu_h(k), \nu_v(k), \kappa_h(k), \kappa_v(k) \)
QNSE results

- Partial scale elimination yields a subgrid-scale model for LES.
- Complete scale elimination yields eddy viscosities and eddy diffusivities for RANS (Reynolds-average Navier-Stokes) models.

In either case, we don’t separate turbulence and waves and treat them as one entity.
Analytical results

We obtain a coupled system of 4 differential equations for scale-dependent, horizontal and vertical eddy viscosities and eddy diffusivities. The system is solved analytically for weak and numerically for arbitrary stratification.

Dashed vertical line shows the threshold of internal wave generation in the presence of turbulence; $k_o = (N^3 / \varepsilon)^{1/2}$ is the Ozmidov wave number.
Weak stable stratification - Limitations of early theories

Lumley-Shur theory predicts an expression for a scale-dependent rate of the spectral energy transfer:

\[ \frac{\epsilon^2}{3} = \epsilon^2/3 \left[ 1 + \left( \frac{k}{k_O} \right)^{-4/3} \right] \]

\( \epsilon \) - rate of the viscous dissipation

“One of the weaknesses in this and other theories is that anisotropy is not adequately addressed” – Holloway (1988)

In the limit of weak stratification, the QNSE theory yields expansion in the powers of the spectral Froude number,

\[ \mathcal{F} \simeq 0.5 \left( \frac{k}{k_O} \right)^{2/3} \]

\[ \begin{align*}
\nu_h/\nu_n &= 1 + 0.095 \mathcal{F}^{-2}, \\
\nu_z/\nu_n &= 1 - 0.31 \mathcal{F}^{-2}, \\
\kappa_h/\nu_n &= \alpha + 0.054 \mathcal{F}^{-2}, \\
\kappa_z/\nu_n &= \alpha - 0.4 \mathcal{F}^{-2},
\end{align*} \]

\[ \nu_n = 0.46 \epsilon^{1/3} k^{-4/3} \]

\( k_O = (N^3/\epsilon)^{1/2} \) is the Ozmidov wavenumber

\( \alpha = Pr_t^{-1} = \nu_n/\kappa_n = 0.72 \) is the inverse turbulent Prandtl number in neutral flows

\( \nu_n \) and \( \kappa_n \) are eddy viscosity and eddy diffusivity in neutral flows
Replacing $\varepsilon$ by $\bar{\varepsilon}$ in the QNSE expression for $\nu_n$ we find:

$$\nu = 0.46\varepsilon^{1/3}k^{-4/3}[1 + (k/k_O)^{-4/3}]^{1/2} \approx \nu_n[1 + 0.5(k/k_O)^{-4/3}]$$

or

$$\frac{\nu}{\nu_n} = 1 + 0.5(k/k_O)^{-4/3}$$

This resembles the QNSE expression for the horizontal viscosity,

$$\frac{\nu_h}{\nu_n} = 1 + 0.4(k/k_O)^{-4/3}$$

This is confusing and intrinsically inconsistent because the Lumley-Shur theory accounts neither for turbulence anisotropy nor for the decrease of the vertical viscosity.

The Lumley-Shur theory predicts a spectral transition from the classical Kolmogorov $k^{-5/3}$ slope to the buoyancy stipulated $k^{-3}$ slope. This transition, however, only takes place for the 1D, vertical spectrum and not for the 3D spectrum.
1D spectra can be calculated using QNSE

\[ E_3(k_1) = \frac{8}{(2\pi)^4} \int U_{33}(\omega, k) d\omega dk_2 dk_3 = 0.626 \varepsilon^{2/3} k^{-5/3} - 0.704 N^2 k_3^{-3} \]

\[ E_3(k_3) = \frac{8}{(2\pi)^4} \int U_{33}(\omega, k) d\omega dk_1 dk_2 = 0.47 \varepsilon^{2/3} k^{-5/3} - 0.143 N^2 k_3^{-3} \]

Vertical spectrum of horizontal velocity

\[ E_1(k_3) = \frac{8}{(2\pi)^4} \int U_{11}(\omega, k) d\omega dk_1 dk_2 = 0.626 \varepsilon^{2/3} k_3^{-5/3} + 0.214 N^2 k_3^{-3} \]

The anisotropization manifests itself as energy increase in the horizontal velocity components at the expense of their vertical counterpart.
Comparison with observational data by Gargett et al., JPO, 1981: “A composite spectrum of vertical shear in the upper ocean”

\[
\frac{E_S}{E_B} = \frac{2k_3^2 E_1(k_3)}{(\epsilon N)^{1/2}} = F(k_3/k_O) = 2 \times 0.626 (k_3/k_O)^{1/3} \left[ 1 + 0.34 (k_3/k_O)^{-4/3} \right]
\]

Gregg, Winkel, Sanford, JPO (1993)

“The results are dramatic, achieving a better collapse than obtained by Gargett et al. The collapse extends across the internal wave range …”
The QNSE theory predicts the vertical spectrum which is in a good agreement with the spectra observed in the stratosphere, troposphere, mesosphere, and thermosphere (our prediction is well approximated by the dashed line).

Another example - atmospheric spectra

S. Smith et al., JAS, 1987
Internal waves and turbulence

The dispersion relation of internal waves with the effect of turbulence:

\[ \omega^2 = N^2 \sin^2 \theta \left\{ 1 - \left( \frac{k}{k_O} \right)^{4/3} \left[ \left( \frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n} \right) \cos^2 \theta + \frac{\left( \frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n} \right) \sin^2 \theta}{4 \sin \theta} \right]^2 \right\} \]

The limit of strong stratification ➔ classical dispersion relation for linear waves, \( \omega = N \sin \theta \). Turbulence dominates at small scales. Criterion for wave generation is \( \omega^2 \geq 0 \) which gives

\[ k_t = k_O \left| \frac{4 \sin \theta}{\left( \frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n} \right) \cos^2 \theta + \left( \frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n} \right) \sin^2 \theta} \right|^{3/2} \approx 32 k_O |\sin \theta|^{3/2} \]

This equation describes a torus
Internal waves exist only for the wavenumbers embedded inside the torus
RANS modeling

Eliminate all fluctuating scales; recast turbulent exchange coefficients as functions of the gradient Richardson number, \( \text{Ri} = N^2/ S^2 \)

\( v_z \) and \( \kappa_z \) decrease with Ri while \( v_h \) and \( \kappa_h \) increase with Ri

Momentum mixing by internal waves ➜

\( v_z \) decreases slower than \( \kappa_z \)

\( v_z \) remains finite even at very large Ri
Stability functions $S_M$ and $S_H$ from the QNSE model and from the Mellor-Yamada model modified by Galperin et al. (1988)

**Temporal mean profile of $\varepsilon$ in the continental shelf seas. Solid line - data; dashed line - an estimate by M-Y level 2 scheme (Rippeth, 2005; Simpson et al. 1996). The M-Y model underpredicts the mixing.**

**QNSE model is not expected to underpredict mixing in strong stable stratification.**
Comparison with data: $Pr_t$ as a function of $Ri$

Both the data and the theory point to the absence of critical $Ri$ and the dependence $Pr_t \propto R_i$ for large $Ri$
Pr_t \propto Ri and \nu_z S^2 - \kappa_z N^2 = \varepsilon \text{ yield } \kappa_z = c \varepsilon N^2 \text{ with } c = 0.25 \text{ to } 0.5 \text{ (in the range by Gregg, Osborn, Weinstock and others).}

The linear dependency, Pr_t \propto Ri, for Ri > 0.2, has been used in some ocean circulation models to facilitate realistic results (Blanke and Delecluse, 1993; Meier, 2001).

All atmospheric and oceanic circulation models employ ‘background’ mixing coefficients in strong stable stratification.

Zhang and Steele, JGR, 2007, Effect of vertical mixing on the Atlantic water layer circulation in the Arctic Ocean:

“Note that a number of vertical mixing schemes have been implemented in general ocean circulation models, including the traditional constant viscosity/diffusivity approach, the Richardson number-dependent scheme [Pacanowski and Philander, 1981], and the Mellor and Yamada [1982] turbulence closure scheme… Mixing below the surface mixed layer is strongly influenced by a “background” diffusivity… Background viscosity is always ten times the background diffusivity, in keeping with the procedure at LANL.”
Critical Richardson number

- Critical Richardson number is a value of Ri at which turbulent mixing ceases, flow becomes re-laminarized and can be described by isotropic molecular viscosity
- Atmospheric and oceanic data show that this rarely happens
- Furthermore, $Pr_t = f(Ri)$ is inconsistent with $Ri_{cr}$
- Yet Reynolds stress models often imply the existence of the critical Ri and underpredict the mixing
- QNSE theory is free of the critical Richardson number because it includes mixing by internal waves and flow anisotropization
- Latest Reynolds stress models (Zilitinkevich et al. (2007), Canuto et al. (2008)) are also designed without $Ri_{cr}$
Monin-Obukhov similarity theory

Monin-Obukhov length scale

\[ L = \frac{u_*^2}{\kappa \beta \theta_*} = \frac{z \alpha_M^2 (Ri)}{Ri \alpha_H (Ri)} \]

\( \zeta = z/L \) can be presented as a function of \( Ri \) and vise versa. Velocity and temperature at the lower grid point are

\[ u_1 = \frac{u_*}{\kappa} \int_{z_0}^{z_1} \frac{dz}{z \alpha_M} \]
\[ \theta_1 = \theta_0 + \frac{\theta_*}{\kappa} \int_{z_0}^{z_1} \frac{dz}{z \alpha_H} \]

The integrals are computed analytically as functions of \( z/L \):

\[ u_1 = \frac{u_*}{\kappa} (\ln \frac{z_1}{z_0} + c_u \frac{z_1}{L} + ...), \quad c_u = 2.25 \]
\[ \theta_1 = \theta_0 + \frac{\theta_*}{1.4 \kappa} (\ln \frac{z_1}{z_0} + c_\theta \frac{z_1}{L} + ...), \quad c_\theta = 2.04 \]
Validation of the QNSE-based RANS models in a single-column formulation in models of atmospheric boundary layers

**K-\(\varepsilon\) format**

- BASE (GABLS 1)
- SHEBA
- CASES-99 (particularly, IOP-9 event)
- CASES-99 (GABLS 2)

**K-\(\ell\) format**

- HIRLAM 7.0 – CASES-99 (GABLS 2)
- January and March, 2005 – fully 3D
- WRF – work in progress
Implementation in HIRLAM

Bias of 2m temperature in simulations of January 2005

Bias of T2M 2 Ident: referj NO: 32
First date: 2005010300 Init. time 00 Length +48

Bias of T2M 2 Ident: nsfja3 NO: 34
First date: 2005010300 Init. time 00 Length +48
Bias of 2m temperature for January, 2005

Red lines – reference HIRLAM, black dots – our model

Skills of the forecast for France

Skills of the forecast for Scandinavia
QNSE vs. Reynolds stress closures

**QNSE**
- $u_i, T$ eqns
- Pressure is solved for exactly
- Rigorous scale elimination
- $K_M, K_H$

**Reynolds stress closures**
- $6 u_i u_j, 3 u_i \theta, \theta^2 \rightarrow 10$ eqns
- Pressure is nonlocal
- Rotta hypothesis
- Closure constants
Conclusions

- Derivation of the QNSE model of turbulence is maximally proximate to first principles; resolves horizontal-vertical anisotropy and accounts for the combined effect of turbulence and internal waves.
- QNSE clarifies early theories of stably stratified turbulence and emphasizes the importance of anisotropy.
- Predicts correct behavior of $Pr_t$ and is free of the critical $\text{Ri}$.
- QNSE-based RANS models have been tested in both $K-\varepsilon$ and $K-\ell$ formats.
- QNSE-based $K-\ell$ model significantly improves turbulence representation in both HI RLAM and WRF.
- QNSE model improves predictive skills of HI RLAM in +24h and +48h weather forecasts and is anticipated to have similar impact in WRF.
- **QNSE-based models are a viable alternative to the Reynolds stress models**
Thank You!
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References


